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The Relation Between Bézout Domain, Elementary Divisor Domain, and Adequate Domain

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Abstract

The aim of this paper is to investigate the relationship between Bézout domain, elementary divisor domain, and adequate domain. A Bézout domain is an integral domain D which every finitely generated ideal of D is principal. An integral domain D is called an elementary divisor domain if every matrix over D is equivalent to Smith normal form matrix. An adequate domain D is a Bézout domain and $RP(a,b)$ exists for all $a, b \in D$ with $a \neq 0$. Here the notion $RP(a,b)$ defined as the relatively prime part of a with respect to b. It is found that every elementary divisor domain is a Bézout domain, but the converse is not true in general. It is shown the sufficient conditions for the Bézout domain being an elementary divisor domain. We also find out that every adequate domain is an elementary divisor domain. Furthermore, every one-dimensional Bézout domain is an adequate domain.

1. INTRODUCTION

Throughout this paper, all rings R are commutative with identity. If F and G are respectively $n \times n$ and $n \times m$ matrices over R, then the system (F, G) is *reachable* if the function from R^{mn} to R^n determined by matrix $[F, F^2G, \dots, F^{n-1}G]$ is surjective and the system (F, G) is *pole assignable* if for every $r_1, r_2, \dots, r_n \in R$, there exists an $m \times n$ matrix K over R such that the characteristic polynomial of the matrix $F - GK$ is $(x - r_1)(x - r_2) \dots (x - r_n)$ [1,2]. In linear systems over commutative rings, a pole assignable system is always reachable. A ring for which the converse holds, i.e., over which every reachable system is pole assignable, is said to have the *pole assignability property*. The term *pole assignability property* can be abbreviated by *PA-property* [3,4]. Rings with this property include the elementary divisor domains and hence all principal ideal domains. Bézout domains are usually better behaved because many important rings may fail to be principal ideal domains.

Investigation of the Bézout domain in linear systems over commutative rings has been developed by many researchers. In theory of regulation of linear systems, [5] gives a general algebraic solution to the problem of regulation of linear systems over arbitrary commutative rings by dynamic output feedback which extends the theory of regulation for such systems. He uses polynomial matrices over the Bézout domain to see its solution.

When we talk about PA-properties, an important problem arises in linear systems over commutative rings, i.e., the PA-property issue for the Bézout domain. [6] have studied the pole placement for reachable system (F, G) over Bézout domains by dynamic output feedback and state-feedback. [1] have also studied it by investigating whether it is an elementary divisor domain.

Besides the Bézout domain, we have another special ring, that is, an adequate ring. [7] introduces this ring as a special Bézout ring with additional property, i.e., satisfies the elementary divisor theorem. A domain with adequate property is called an adequate domain. Same with Bézout domain, whether the adequate domain has PA-property? To verify that it has PA-property, we only have to investigate whether it is an elementary divisor domain.

In relation to the above results, we are interested to see The Relationship between Bézout domain, elementary divisor domain and adequate domain. However, we only investigate the algebraic relationship between them without seeing those PA-properties since it immediately follows from the elementary divisor domain.

2. METHOD

To support our research, we collect some literature such as journals, books and theses that are relevant for this topic. Before the results of our research are presented, we need to give some preliminaries about the Bézout domain, elementary divisor domain and adequate domain that consist of definitions and some properties.

2.1 Bézout Domain and Its Properties

A ring R is a Bézout ring, if every finitely generated ideal is principal [8]. For domains, the notion of Bézout is equivalent.

Definition 2.1.1. The Bézout domain is an integral domain D which every finitely generated ideal of D is principal. More precisely, an integral domain D is called Bézout domain if for each $X \subseteq D, |X| < \infty$ satisfies $\langle X \rangle = \langle d \rangle$, for some $d \in D$.

It is clear that every principal ideal domain is a Bézout domain. However, the converse is not true in general. In [9], one example of a Bézout domain that is not a principal ideal domain is also mentioned, that is the ring of all entire functions on the complex plane.

Based on the ideal property in ring theory that every finitely generated ideal is principal if and only if every ideal with two generators is principal, then the first property of the Bézout domain is obtained as follows.

Theorem 2.1.2. [10] *An integral domain D is a Bézout domain if and only if each pair of elements a and b in D have a greatest common elements(gcd) $d \in D$ that is a linear combination of a and b such that $d = ax + by$, for some $x, y \in D$.*

Proof If D is a Bézout domain then the finitely ideal $\langle a, b \rangle$ is principal. Suppose that d is a generator of ideal $\langle a, b \rangle$, whence there exist $x, y \in D$ with $ax + by = d$. Clearly d is a gcd of a and b. Conversely, suppose D has a gcd algorithm of the type describe, and that I is an ideal of D that generated by $a_1, a_2, \dots , a_n$. Let d be a GCD of a_1 and a_2. Then by definition of gcd, we have $a_1 \in \langle d \rangle$ and $a_2 \in \langle d \rangle$ whence $I \subseteq \langle d, a_3, \dots, a_n \rangle$. Since also we have $d = a_1 x + a_2 y$ for some $x, y \in D$, we get reverse inclusion and D can be generated by $n - 1$ elements. By descent on n, we deduce that I is principal and hence that D is a Bézout domain.

Lemma 2.1.3. [2] *Let D be an integral domain. A nonzero ideal I of D is free as a D-module if and only if I is principal.*

Proof Let I is a nonzero ideal of D. We can say that I is a submodule of D, so $rank(I) \leq rank(D) = 1$. Then either $rank(I) = 0$ or $rank(I) = 1$. Since D is an integral domain, $rank(I) = 0$ implies $I = \{0\}$. This contradicts the assumption and hence $rank(I) = 1$. It follows from the hypothesis that I is free as a D-module and $rank(I) = 1$, we get $I = \langle u \rangle$ for some $u \in D$. Conversely, assume that a nonzero ideal I of D is principal. Then, $I = \langle r \rangle$, for some $r \in D$. If $r = 0$, then $r = \{0\}$ is free as a D-module. If $r \neq 0$, then the mapping $r \mapsto rx$ is a D-module isomorphism from D to I. So, I is free as a D-module.

From Lemma 2.1.3 we can declare the following other characteristics of the Bézout domain.

Theorem 2.1.4. [2] *Let D be an integral domain. Then D is a Bézout domain if and only if each finitely generated submodule of a free D-module is free.*

Proof Suppose that D is a Bézout domain. If each finitely generated submodule of a free D-module is free, then each finitely generated ideal I of D is free. It follows that I is principal, so D is a Bézout domain. Conversely, if D is a Bézout domain, then each finitely generated ideal of D is a free D-module. Let A be finitely generated submodule of the free D-module F. If $\{x_i\}$ is a basis for F, then since the finitely many generators of A involve only finitely many of the x's, it follows that $A \subseteq Dx_1 \oplus \ldots \oplus Dx_n$ for some positive integer n. To prove that A is free, we make induction on n. If $n = 1$, then $A \subseteq Dx_1 \cong D$ and therefore A is isomorphic to finitely generated ideal of D. For the induction step, consider the map $\phi : A \to D$ defined as follows: each element $a \in A$ can be written uniquely in the form $a = r_1x_1 + r_2x_2 + \ldots + r_nx_n$, for elements $r_i \in D$. Set $\phi(a) = r_n$. Then ϕ is evidently a D-homomorphism and consequently $\phi(A)$ is a D-submodule of D, that is an ideal of D. Since A is finitely generated, $\phi(A)$ is finitely generated and therefore free as a D-module. This all gives rise to the following exact sequence

$$0 \to A \cap (Dx_1 \oplus \ldots \oplus Dx_{n-1}) \to A \to \phi(A) \to 0$$

where A is free and hence projective. Consequently, this sequence splits and we have that

$$A \cong \phi(A) \oplus [A \cap (Dx_1 \oplus \ldots \oplus Dx_{n-1})].$$

But $A \cap (Dx_1 \oplus \ldots \oplus Dx_{n-1})$ is finitely generated, being a homomorphic image of A, and is also a submodule of a free D-module on $(n-1)$ generators. By the induction assumption, $A \cap (Dx_1 \oplus \ldots \oplus Dx_{n-1})$ is free and so is A.

2.2 Elementary Divisor Domain and Its Properties

In this section, we give definition and some properties about the elementary divisor domain.

Definition 2.2.1. [11,12] If every matrix over ring R admits diagonal reduction then R is called an elementary divisor ring . Specifically, R is an elementary divisor ring if every matrix $A \in M_{m\times n}(R)$, there exist invertible matrices $P \in GL_m(R)$ and $Q \in GL_n(R)$ such that $PAQ = D_A$, where $D_A = [d_{i,j}]$ is diagonal matrix and $d_{i,i} | d_{i+1,i+1}$ for every i . The matrix D_A is called *Smith normal form* of A. In this case, matrix A is equivalent to D_A. A domain D is an elementary divisor domain if every matrix over D is equivalent to Smith normal form matrix.

In [13], the term *Henriksen elementary divisor ring* is the same as elementary divisor ring.

Lemma 2.2.2. [7] *If all* 2×1 *and* 2×2 *matrices over* R *admit diagonal reduction then all matrices over* R *also admit diagonal reduction and therefore* R *is an elementary divisor ring.*

Proof Let A be an $m \times n$ matrix. It suffices to show for the case $m \geq n$. By induction, suppose the lemma is true for smaller m and for the given m if n is smaller. From the hypothesis, we prove for m is at least 3. Write A as a block matrix, that is, A_1 as the first row and the remaining $m-1$ rows with A_2. Since A_2 is a small dimensional matrix, we can find invertible matrix P_1, Q_1 such that $B = P_1A_2Q_1 = diag(x, \ldots)$ where B is a Smith normal form matrix. Note that

$$C = \begin{bmatrix} 1 & 0 \\ 0 & P_1 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} Q_1 = \begin{bmatrix} A_1Q_1 \\ P_1A_2Q_1 \end{bmatrix} = \begin{bmatrix} A_1Q_1 \\ B \end{bmatrix}.$$

Now, let D as the first rows of C and E for the remaining rows. Applying induction again, we can find invertible matrices P_2, Q_2 such that $F = P_2DQ_2 = diag(y, \ldots)$ where F is a Smith normal form matrix. Note that

$$H = \begin{bmatrix} P_2 & 0 \\ 0 & I_{m-2} \end{bmatrix} \begin{bmatrix} D \\ E \end{bmatrix} Q_2 = \begin{bmatrix} P_2DQ_2 \\ EQ_2 \end{bmatrix} = \begin{bmatrix} F \\ EQ_2 \end{bmatrix}.$$

It is easy to see that y divides all entries of F and since $D = {P_2}^{-1}F{Q_2}^{-1}$, y is also a divisor of all elements of D. Since $B = diag(x, \ldots)$, then y is also divisor of x. All elements of EQ_2 are linear combinations of all elements of E and hence they are divisible by both x and y. Thus, y divides all entries of H.
Elementary operations are used to eliminate the first column of H and we get

$$\begin{bmatrix} y & 0 \\ 0 & K \end{bmatrix},$$

where y is also divisor of all elements of K. Applying the inductive hypothesis, the reduction is complete.

The next theorem gives us a necessary and sufficient condition for R being an elementary divisor ring.

Theorem 2.2.3. [14] *R is an elementary divisor ring if and only if all 2×2 matrix over R admit diagonal reduction.*

Proof Suppose that R is an elementary divisor ring. Let A be an 2×2 matrix over R. It follows from definition, A admits diagonal reduction. Conversely, suppose that all matrix 2×2 matrix over R admit diagonal reduction. Based on Lemma 2.2.2., we need only to prove that all 2×1 matrix over R also admit diagonal reduction. Let a, b are arbitrary elements in R. We construct a matrix F such that

$$F = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}.$$

Then, by hypothesis, the matrix F admit diagonal reduction. Consequently, F is a direct sum of cyclic modules and therefore R is a Hermite ring. It follows from the definition of the Hermite ring, every 2×1 matrix over R admit diagonal reduction. The proof is complete.

Recall that a principal ideal domain is an integral domain whose every ideal is principal. The next theorem gives us a relationship between principal ideal domain and elementary divisor domain.

Theorem 2.2.4 *Every principal ideal domain is an elementary divisor domain.*

Proof It is enough to show that every principal ideal ring is an elementary divisor ring [15]. Assume that R is a principal ideal ring. R can be written as a finite direct sum $R = R_1 \oplus R_2 \oplus \dots \oplus R_s$ where each R_i is either a principal ideal domain or a special principal ideal ring. In either case, $Z(R_i) \subseteq J(R_i)$ where $Z(R_i)$ and $J(R_i)$ denote the set of zero divisors of R_i and the Jacobson radical of R_i respectively. Consequently, that implies R_i is a Hermite ring. Since each R_i is principal ideal domain then each R_i is Noetherian. Recall that every Noetherian and Hermite ring is an elementary divisor ring. In other word, each R_i is an elementary divisor ring. Furthermore, we need only to show that the finite direct sum of elementary divisor rings is again elementary divisor rings. Choose $m, n \geq 1$ and let $A \in M_{m\times n}(R)$. Since $R = R_1 \oplus R_2 \oplus \dots \oplus R_s$ then $A = A_1 + A_2 + \dots + A_s$ with each $A_i \in M_{m\times n}(R_i)$ for $i = 1,2, \dots, s$. Since each R_i is elementary divisor ring, each A_i admits diagonal reduction, i.e., there exist $P_i \in GL_m(R_i)$ and $Q_i \in GL_n(R_i)$ such that $D_i = P_i A_i Q_i$ with $D_i = diag(d_{1i}, d_{2i}, \dots, d_{ti})$ and $d_{1i} | d_{2i} | \dots | d_{ti}$ for $i = 1,2, \dots, s$. Note that $r = \min\{m, n\}$. Set $P = P_1 + P_2 + \dots + P_s$ and $Q = Q_1 + Q_2 + \dots + Q_s$. Since $R_i R_j = \langle 0 \rangle$ whenever $i \neq j$, then P and Q are invertible with inverses $P^{-1} = {P_1}^{-1} + {P_2}^{-1} + \dots + {P_s}^{-1}$ and $Q^{-1} = {Q_1}^{-1} + {Q_2}^{-1} + \dots + {Q_s}^{-1}$. This implies that $P \in GL_m(R)$ and $Q \in GL_n(R)$. Moreover,

$$\begin{aligned} PAQ &= (P_1 + P_2 + \dots + P_s)A(Q_1 + Q_2 + \dots + Q_s) \\ &= P_1AQ_1 + P_2AQ_2 + \dots + P_sAQ_s \\ &= diag\left(\sum_{i=1}^{s} d_{1i}, \sum_{i=1}^{s} d_{2i}, \dots, \sum_{i=1}^{s} d_{ti}\right), \end{aligned}$$

and $\sum_{i=1}^{s} d_{1i} | \sum_{i=1}^{s} d_{2i} | \dots | \sum_{i=1}^{s} d_{ti}$. Thus, A admits diagonal reduction. Finally, $R = R_1 \oplus R_2 \oplus \dots \oplus R_s$ is an elementary divisor ring.

Using the above theorem, we can say that the example of an elementary divisor domain is any principal ideal domain. However, there is an example that an elementary divisor domain is not a principal ideal domain. It is a ring $H(\Omega)$ of all complex holomorphic in an open connected set $\Omega \subseteq \mathbb{C}$ [16].

2.3 Adequate Domain and Its Properties

As previously stated in introduction, an adequate ring is a special Bézout ring. A more precise definition is given below.

Definition 2.3.1. [17] A ring R is said an adequate ring if R is a Bézout ring and for all a, b in R with $a \neq 0$, $RP(a, b)$ exists.

The notion of $RP(a, b)$ defined as *the relatively prime part* of a with respect to b. It means that for every a, b in R with $a \neq 0$, there exists two nonzero elements r, s in R such that $a = rs$, $\gcd(r, b) = 1$ and $\gcd(t, b) \neq 1$ for any non-unit factor t of s. It is easy to verify that the relatively prime property is equivalent with for every a, b in R with $a \neq 0$, there exists two nonzero elements r, s in R such that $a = rs, rR + bR = R$ and $tR + bR \neq R$ for any non-unit factor t of s [18]. Regular (commutative with identity) ring [19] and local ring [18] are examples of adequate ring. An integral domain that satisfies adequate property is called an adequate domain.

Similar to the Bézout domain and elementary divisor domain, we have the following theorem.

Theorem 2.3.2 *Every principal ideal domain is an adequate domain.*

Proof Suppose that R is a principal ideal domain. We will show that R is an adequate domain. First, we will verify that R is a Bézout domain. Since R is a principal ideal domain, every ideal is principal. Hence, every finitely generated ideal is principal. This proves that R is a Bézout domain.
In the last step, we will prove that relatively prime part property holds for all a, b in R with $a \neq 0$. Note that the arithmetic fundamental theorem holds for any principal ideal domain. Hence, it guarantees the existence of $RP(a, b)$ for $a \neq 0$.

Using the above theorem, we can say that the example of an adequate domain is any principal ideal domain. However, the example of an adequate ring which is not a principal ideal domain is the set of integral function with coefficient in a field F [20].

3. RESULTS AND DISCUSSION

From this section, the structure of the ring is an integral domain D, i.e., a commutative ring with identity whose all elements are not zero divisors.

First, the relationship between the Bézout domain and the elementary divisor domain is shown as in the following theorem.

Theorem 3.1.1. *Every elementary divisor domain is a Bézout domain.*

Proof Let D is an elementary divisor domain and $a, b \in D$. Based on Theorem 2.2.3, suppose A be the 2×2 diagonal matrix with a and b on the diagonal. Then there are invertible matrices $P, Q \in GL_2(R)$ such that PAQ is in Smith normal form. Writing these multiplications out explicitly, we see that

$$PAQ = \begin{bmatrix} d & 0 \\ 0 & e \end{bmatrix} = \begin{bmatrix} r_1 & s_1 \\ t_1 & u_1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} r_2 & s_2 \\ t_2 & u_2 \end{bmatrix} = \begin{bmatrix} r_1 a r_2 + s_1 b t_2 & r_1 a s_2 + s_1 b u_2 \\ t_1 a r_2 + u_1 b t_2 & t_1 a s_2 + u_1 b u_2 \end{bmatrix}.$$

Therefore, d is a linear combination of a and b, so $d \in \langle a, b \rangle$.
Since $d|e$, we can write $e = df$. Then $A = P^{-1}(PAQ)Q^{-1}$ can be written

$$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} r_3 & s_3 \\ t_3 & u_3 \end{bmatrix} \begin{bmatrix} d & 0 \\ 0 & df \end{bmatrix} \begin{bmatrix} r_4 & s_4 \\ t_4 & u_4 \end{bmatrix} = \begin{bmatrix} r_3 d r_4 + s_3 d f t_4 & r_3 d s_4 + s_3 d f u_4 \\ t_3 d r_3 + u_3 d f t_2 & t_3 d s_4 + u_3 d f u_4 \end{bmatrix}.$$

This means that both a and b are multiples of d, so $a, b \in \langle d \rangle$. Therefore $\langle a, b \rangle = \langle d \rangle$ and based on Theorem 2.1.2 then D is a Bézout domain.

However, the converse of Theorem 3.1.1 is not true in general. Let $X = R^+ \cup S^+$ where $R^+ = \{(x,0): x \geq 0\} \subseteq \mathbb{R}^2$ and $S^+ = \{(x, \sin \pi x): x \geq 0\} \subseteq \mathbb{R}^2$ then the ring of all real-valued continuous function $C(\beta(X) - X)$, where $\beta(X)$ denotes the *Stone-Čech compactification* of X, is a Bézout ring but not an elementary divisor ring [21]. It follows from this fact, we have to add some conditions to the Bézout domain for being an elementary divisor domain. We have the first condition below.

Theorem 3.1.2. [1,2] *Let D be a Bézout domain having only countably many maximal ideals. Then D is an elementary divisor domain.*

Proof It suffices to prove that each matrix of the form $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $\gcd(a,b,c,d) = 1$ can be diagonalized. Let M be a maximal of D. We claim that A is equivalent to a matrix $\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$ where a' divides a and $a' \notin M$. Write $\gcd(a,c) = pa + qc$, for some $p, q \in D$, then

$$\begin{bmatrix} p & q \\ \frac{c}{\gcd(a,c)} & \frac{c}{\gcd(a,c)} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \gcd(a,c) & * \\ 0 & * \end{bmatrix}.$$

Since the determinant of the matrix on left side is -1 which is -1 is a unit in D, then the matrix is invertible. Thus, A is equivalent to a matrix $\begin{bmatrix} a^* & b^* \\ c^* & d^* \end{bmatrix}$ where $a^* = \gcd(a,c)$ divides a. If $a^* \notin M$, it is done.
Suppose that $b^* \notin M$ and $\gcd(a^*, b^*) = xa^* + yb^*$, for some $x, y \in D$ then

$$\begin{bmatrix} a^* & b^* \\ c^* & d^* \end{bmatrix} \begin{bmatrix} x & - \\ y & - \end{bmatrix} = \begin{bmatrix} \gcd(a^*, b^*) & - \\ - & - \end{bmatrix}$$

where $\gcd(a^*, b^*)$ divides a^*. Therefore $\gcd(a^*, b^*)$ divides a and $\gcd(a^*, b^*) \notin M$. If $b^* \in M$ then

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a^* & b^* \\ c^* & d^* \end{bmatrix} = \begin{bmatrix} a^* + c^* & b^* + d^* \\ c^* & d^* \end{bmatrix} = \begin{bmatrix} a^* & b^* + d^* \\ 0 & d^* \end{bmatrix},$$

with $c^* = 0$. This returns us to the case just treated and the claim is justified.
Let $\{M_1, M_2, \ldots\}$ be the set of all maximal ideals of D. Do the above business to $M_1, M_2, \ldots$ in succession obtaining elements $a_1, a_2, \ldots \in D$ such that $\langle a \rangle \subseteq \langle a_1 \rangle \subseteq \langle a_2 \rangle \subseteq \cdots$. Therefore $\cup_{i=1}^{\infty} \langle a_i \rangle$ is contained in no maximal ideal of D and so $\cup_{i=1}^{\infty} \langle a_i \rangle = D$. It follows that some a_n is a unit of D and hence that A is equivalent to a matrix $\begin{bmatrix} a_n & - \\ - & - \end{bmatrix}$ where a_n is a unit. Such a matrix can easily be diagonalized then this prove that D is an elementary divisor domain.

The next theorem gives the relationship between adequate and elementary divisor domain.

Theorem 3.1.3. *Every adequate domain is an elementary divisor domain.*

Proof We use Theorem 2.2.3 to verify this theorem. Suppose D is an adequate domain. We will show that every 2×2 matrix over D is equivalent to a Smith normal form matrix. Let $A \in M_{2\times 2}(D)$ with $A = [a_{ij}]$ for $i, j = 1,2$. If rank of A is $r = 0$ then $A = O$ where O is a zero matrix and hence A is a Smith normal form matrix. If $r = 1$, then two rows (or columns) of A are linearly dependent in R. If we let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

then there are two relatively prime elements $p, q \in D$ such that $pa_{11} + qa_{12} = 0$ and $pa_{21} + qa_{22} = 0$. Since p and q are relatively prime then $\gcd(p, q) = 1$ and hence there exist $u, v \in D$ such that $up + vq = 1$. If we construct a matrix

$$\begin{bmatrix} v & -u \\ p & q \end{bmatrix},$$

then the product of this matrix and A yields

$$\begin{bmatrix} v & -u \\ p & q \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} va_{11} - ua_{21} & va_{12} - ua_{22} \\ pa_{11} + qa_{21} & pa_{12} + qa_{22} \end{bmatrix} = \begin{bmatrix} p' & q' \\ 0 & 0 \end{bmatrix} = A',$$

where $p' = va_{11} - ua_{21}$ and $q' = va_{12} - ua_{22}$. Since D is an adequate domain, there exist two elements $d, p_1 \in D$ such that $p' = dp_1$, $\gcd(d, q') = 1$, and $\gcd(t, q') \neq 1$ for any non-unit factor t of p_1. If p_1 is not a unit, then $\gcd(p_1, q') = f \neq 1$, and hence $p_1 = ft$, $q' = fq''$ and $\gcd(t, q'') = 1$. Since $\gcd(d, q') = 1$ we have $\gcd(d, q'') = 1$, so $\gcd(dt, q'') = 1$. Thus, there exist two elements $l, m \in D$ such that $ldt + mq'' = 1$. Set the matrix

$$C = \begin{bmatrix} l & -q'' \\ m & dt \end{bmatrix}.$$

This matrix have determinant 1, so it is invertible. Moreover,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} p' & q' \\ 0 & 0 \end{bmatrix}\begin{bmatrix} l & -q'' \\ m & dt \end{bmatrix} = \begin{bmatrix} dft & fq'' \\ 0 & 0 \end{bmatrix}\begin{bmatrix} l & -q'' \\ m & dt \end{bmatrix} = \begin{bmatrix} dlft + fq''m & 0 \\ 0 & 0 \end{bmatrix}.$$

and $IA'C$ is a Smith normal form matrix, because $IA'C$ is a diagonal matrix and $(dlft + fq''m)|0$.

If $rank(A) = 2$ and let $\gcd(a_{ij}) = v$, then there exist $b_{ij} \in D$ such that $a_{ij} = vb_{ij}$ with $\gcd(b_{ij}) = 1$ for every $i, j = 1,2$. Thus, we can write A as follows.

$$A = vB = v\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}.$$

Helmer [17] states that there exists $y \in D$ such that $\gcd(b_{11}t + b_{21}, b_{12}t + b_{22}) = 1$, so that there are $r, s \in D$ such that $(b_{11}t + b_{21})r + (b_{12}t + b_{22})s = 1$. Construct two matrices $P, Q \in M_{2\times 2}(D)$

$$P = \begin{bmatrix} t & 1 \\ (rb_{11} + sb_{12})t - 1 & rb_{11} + sb_{12} \end{bmatrix}$$

and

$$Q = \begin{bmatrix} r & -(b_{12}t + b_{22}) \\ s & b_{11}t + b_{21} \end{bmatrix}.$$

Both matrices have determinant 1, and hence they are invertible. Since $A = vB$, then $PAQ = vPBQ$ and by direct computation, we get

$$\begin{aligned} dPBQ &= d\begin{bmatrix} t & 1 \\ (rb_{11} + sb_{12})t - 1 & rb_{11} + sb_{12} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\begin{bmatrix} r & -(b_{12}t + b_{22}) \\ s & b_{11}t + b_{21} \end{bmatrix} \\ &= d\begin{bmatrix} tb_{11} + b_{21} & tb_{12} + b_{22} \\ rb_{11}^2t + sb_{12}b_{11}t - b_{11} + rb_{11}b_{21} + sb_{12}b_{21} & rb_{11}b_{12}t + sb_{12}^2t - b_{12} + rb_{11}b_{22} + sb_{12}b_{22} \end{bmatrix} \\ &\quad \begin{bmatrix} r & -(b_{12}t + b_{22}) \\ s & b_{11}t + b_{21} \end{bmatrix} \\ &= d\begin{bmatrix} tb_{11} + b_{21} & tb_{12} + b_{22} \\ (rb_{11} + sb_{12})(b_{11}t + b_{21}) - b_{11} & (rb_{11} + sb_{12})(b_{12}t + b_{22}) - b_{12} \end{bmatrix}\begin{bmatrix} r & -(b_{12}t + b_{22}) \\ s & b_{11}t + b_{21} \end{bmatrix} \end{aligned}$$

$$= d\begin{bmatrix} 1 & 0 \\ (rb_{11}+sb_{12})\big((b_{11}t+b_{21})r+(b_{12}t+b_{22})s\big)-b_{11}r-b_{12}s & b_{11}b_{22}-b_{12}b_{21} \end{bmatrix}$$
$$= d\begin{bmatrix} 1 & 0 \\ (rb_{11}+sb_{12})-b_{11}r-b_{12}s & b_{11}b_{22}-b_{12}b_{21} \end{bmatrix}$$
$$= d\begin{bmatrix} 1 & 0 \\ 0 & b_{11}b_{22}-b_{12}b_{21} \end{bmatrix}.$$

Since the last matrix is a diagonal matrix and $d|d(b_{11}b_{22}-b_{12}b_{21})$, then it is a Smith normal form matrix. Hence, we are done for $rank(A)=2$. The proof is complete.
Similar with Theorem 3.1.1, the converse of Theorem 3.1.3 is not true in general. An integral domain D whose group of divisibility $G(D)$, where $G(D)$ is the multiplicative group of nonzero principal fractional ideals of D, is an elementary divisor domain but not an adequate domain. For details, see [22].

Let R be a commutative ring with identity and let n be a positive integer. If there exists a chain
$$P_0 \subset P_1 \subset \cdots \subset P_n$$
of prime ideals of R, and there is no chain longer than that, then R has *dimension* n or R has *Krull dimension* n. If there is no bound to the lengths of chains of prime ideals of R, then R is *infinite dimensional*. For example, the ring of integer $\mathbb{Z}$ have dimension 1; the polynomial ring $\mathbb{Z}[x_1,x_2,\dots,x_n]$ has dimension $n+1$ and if K is a field then the polynomial ring $K[x_1,x_2,\dots,x_n]$ has dimension n [2].

Theorem 3.1.4 [1,2] *Let D is a one-dimensional Bézout domain. Then D is an adequate domain.*

Proof Suppose that D is a one-dimensional Bézout domain. We will show that D is an adequate domain. Consider the following sequence of elements of D

$$b_1=\frac{b}{\gcd(b,d)},\ b_2=\frac{b_1}{\gcd(b_1,d)},\ b_3=\frac{b_2}{\gcd(b_2,d)},\dots,b_n=\frac{b_{n-1}}{\gcd(b_{n-1},d)},\dots$$

We claim that for some positive integer n, $\gcd(b_n,d)=1$. Otherwise, consider the following chain of ideals of D

$$\langle b,d\rangle \subseteq \langle b_1,d\rangle \subseteq \langle b_2,d\rangle \subseteq \cdots$$

The union $\langle b,d\rangle \cup (\cup_{i=1}^{\infty}\ \langle b_i,d\rangle)$ is a proper ideal of D and thus is contained in a maximal ideal M of D. Since D is a Bézout domain, then it is a Prüfer domain and so it is a "valuation domain" D_M, i.e., an integral domain with property all ideals are linearly ordered or totally ordered [23]. Hence, for every i, the ideals $\langle b_i\rangle D_M$ and $\langle d\rangle D_M$ can be compared under inclusion $\subseteq$. So, it is either $\langle b_i\rangle D_M \subseteq \langle d\rangle D_M$ or $\langle d\rangle D_M \subseteq \langle b_i\rangle D_M$. It is easy to see that for every i, $\langle b_i,d\rangle D_M=\langle b_i\rangle D_M$ or $\langle b_i,d\rangle D_M=\langle d\rangle D_M$. In the valuation domain D_M, if $\langle b_i,d\rangle D_M=\langle b_i\rangle D_M$ for some i, then $\langle b_i,d\rangle D_M=D_M$, a contradiction. The only option is that $\langle b_i,d\rangle D_M=\langle d\rangle D_M$ for each i, but this implies $b\in\langle d^i\rangle D_M$ for each i and hence $b=0$ since D_M is one-dimensional. The claim has been proved.
Now, set $b_n=r$ and $s=\frac{b}{r}$. We have $\gcd(r,d)=1$, $b=rs$ and the only exercise for us to verify that no non unit factor of s is relatively prime to d. Since $s=\frac{b}{r}$ then $s=\gcd(b,d)\gcd(b_1,d)\dots\gcd(b_{n-1},d)$. If $s=uv$ for $u,v\in D$ and u is not a unit, let M_u be a maximal ideal of D containing u. By property of ideal, $s\in M_u$ and so $\langle b_i,d\rangle \subseteq D_M$ for some i. Thus, $u\in M_u$ and $d\in M_u$. Consequently, $\langle u,d\rangle\subseteq M_u$. It means that u and d is not relatively prime since $M_u\neq D$.
Combining Theorem 3.1.3 and Theorem 3.1.5, we have the following consequence that gives the second condition to the Bézout domain for being an elementary divisor domain.
Corollary 3.1.5 *Let D is a one-dimensional Bézout domain. Then D is an elementary divisor domain.*

4. CONCLUSIONS

We can summarize all of our main results into the following five points.

1. If D is an adequate domain, then it is an elementary divisor domain and hence a Bézout domain.
2. If D be a Bézout domain having only countably many maximal ideals, then D is an elementary divisor domain.
3. If D is a one-dimensional Bézout domain, then D is an elementary divisor domain.
4. If $X = R^+ \cup S^+$ where $R^+ = \{(x, 0): x \geq 0\} \subseteq \mathbb{R}^2$ and $S^+ = \{(x, \sin \pi x): x \geq 0\} \subseteq \mathbb{R}^2$ then the ring of all continuous real-valued function $C(\beta(X) - X)$ where $\beta(X)$ denotes the *Stone-Čech compactification* of X, is a Bézout ring but not an elementary divisor ring.
5. An integral domain D whose group of divisibility $G(D)$, where $G(D)$ is the multiplicative group of nonzero principal fractional ideals of D, is an elementary divisor domain but not an adequate domain.

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REFERENCES

[1] J.W. Brewer, C. Naude, and G. Naude, "On Bézout domains, elementary divisor rings, and pole assignability", *Communications in Algebra*, vol. 12, no. 24, pp. 2987-3003, 1984. https://doi.org/10.1080/00927878408823140
[2] J.W. Brewer, J.W. Bunce, and F.S. Van Vleck, *Linear Systems over Commutative Rings*, New York: Marcel Dekker, Inc., 1986.
[3] J.W. Brewer, D. Katz, and W. Ullery, "On the pole assignability property over commutative rings", *Journal of Pure and Applied Algebra*, vol. 48, pp. 1-7, 1985. https://doi.org/10.1016/0022-4049(87)90103-4
[4] J.W. Brewer, D. Katz, and W. Ullery, "Pole assignability in polynomial ring, power series rings, and Prüfer domains", *Journal of Algebra*, vol. 106, pp. 265-286, 1987. https://doi.org/10.1016/0021-8693(87)90031-7
[5] E. Emre, "Regulation of linear systems over rings by dynamic output feedback", *Systems & Control Letters*, vol. 3, pp. 57-62, 1983. https://doi.org/10.1016/0167-6911(83)90039-7
[6] E. Emre and P.P. Khargonekar, "Pole placement for linear systems over Bézout domains", *IEEE Transactions on Automatic Control*, vol. AC-29, no. 1, pp. 90-91, 1984. https://doi.org/10.1109/TAC.1984.1103371
[7] I. Kaplansky, "Elementary divisors and modules", *Transactions of the American Mathematical Society*, vol. 66, pp. 464-491, 1949. https://doi.org/10.1090/S0002-9947-1949-0031470-3
[8] B. Zabavsky and A. Gatalevych, "A commutative Bézout domain *PM** domain is an elementary divisor ring", *Algebra and Discrete Mathematics*, vol. 19, no. 2, pp. 295-301, 2015.
[9] M.D. Walton, *Bézout Domains and Elementary Divisor Domains: Are They the Same?* M.Sc. [Theses]. Utah: BYU, 2023. Available: BYU ScholarsArchive.
[10] D.S. Dummit and R.M. Foote, *Abstract Algebra,* Third Edition, New Jersey: John Wiley & Sons, Inc., 2004.
[11] H. Chen and M.S. Abdolyousefi, "Elementary matrix reduction over Bézout domains", *Journal of Algebra and Its Applications*, vol. 18, no. 18, pp. 1-12, 2019. https://doi.org/10.1142/S021949881950141X
[12] D. Lorenzini, "Elementary divisor domains and Bézout domains", *Journal of Algebra*, vol. 371, pp. 609-619, 2012. https://doi.org/10.1016/j.jalgebra.2012.08.020
[13] B.V. Zabavsky, O.V. Domsha, and O.M. Romaniv, "Clear rings and clear elements", *Matematychni Studii,* vol. 55, no. 1, pp. 3-9, 2021. https://doi.org/10.30970/ms.55.1.3-9
[14] M.X. Larsen, W.J. Lewis, and T.S. Shores, "Elementary divisor rings and finitely presented modules", *Transactions of the American Mathematical Society,* vol. 187, pp. 231-248, 1974. https://doi.org/10.2307/1997051
[15] W.C. Brown, *Matrices over Commutative Rings*, New York: Marcel Dekker, Inc., 1993.
[16] C. Caldeira and J.F. Queiró, "Invariant factor of products over elementary divisor domains", *Linear Algebra and its Applications*, vol. 485, pp. 345-358. https://doi.org/10.1016/j.laa.2015.07.038

[17] O. Helmer, "The elementary divisor theorem for certain rings without chain condition", *Bulletin of the American Mathematical Society*, vol. 49, pp. 225-236, 1943. https://doi.org/10.1090/S0002-9904-1943-07886-X
[18] N. Mahdou and M. Zennayi, "On adequate rings", *Journal of Taibah University for Science*, vol. 9, pp. 320-325, 2015. https://doi.org/10.1016/j.jtusci.2015.01.006
[19] L. Gillman and M. Henriksen, "Some remarks about elementary divisor rings", *Transactions of the American Mathematical Society*, vol. 82, pp. 362-365, 1956. https://doi.org/10.1090/S0002-9947-1956-0078979-8
[20] M. Zennayi, "On adequate rings: the basic properties", *Gulf Journal of Mathematics*, vol. 3, no. 3, pp. 48-56, 2015. https://doi.org/10.56947/gjom.v3i3.147
[21] L. Gillman and M. Henriksen, "Rings of continuous functions in which every finitely generated ideal is principal", *Transactions of the American Mathematical Society*, vol. 82, no. 2, pp. 366-391, 1956. https://doi.org/10.2307/1993054
[22] J.W. Brewer, P.F. Conrad, and P.R. Montgomery, "Lattice-ordered groups and a conjecture for adequate domains", *Proceedings of the American Mathematical Society*, vol. 43, no. 1, pp. 31-35, 1974. https://doi.org/10.2307/2039319
[23] R. Gilmer, *Multiplicative Ideal Theory*, New York: Marcel Dekker, Inc., 1972.