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# Probability Generating Functions: Theory and Applications to Distribution Generation, Branching Processes and Queueing Model

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**Abstract**

The probability generating function (PGF) of a discrete random variable is a concise way to describe the corresponding probability distribution and facilitate analysis. This paper will revisit and explore certain key properties of the PGFs, focusing on structural properties, moment characterization and stability under convolution. Through applications to distribution generation, branching processes, and queueing models, PGFs are shown to provide clear insight into extinction probabilities and steady-state behavior. The results confirm that probability generating functions provide a coherent and useful framework for both theory and applications in discrete stochastic modelling.

## 1. INTRODUCTION

Probability generating functions (PGFs) play an essential role in probability theory and constitute a convenient and synthesized framework for representing discrete random variables. The use of PGFs (probability generating functions) allows you to treat complex stochastic phenomenon (e.g. earthquakes, epidemics, climate change) in depth which otherwise would be analytically cumbersome [1]. By the ease with which we can change the subject only the dependent variable can be expressed in terms of anyone. As a result, PGFs have become essential tools in a variety of scientific and engineering domains, including biological population dynamics, epidemiology and the performance analysis of communication and service networks [2]. One value of PGFs is that they make probabilistic manipulations in the form of algebraic manipulations. The analytic properties of generating functions make it possible to carry out operations such as computing moments, deriving sums of independent random variables, and investigating limiting distribution [1]. By applying the PGFs as the necessary tools that help in bridging intuitive reasoning with formal mathematics, the usefulness of PGFs is observed in both theory and pragmatics.

In discrete stochastic modeling, PGF serves as the mainstay of discrete states to analyze various systems/effects which happen at discrete time. Notable examples are branching processes, which describe population growth, extinction probability and epidemic spread, and queueing systems, which are the building blocks to study waiting line behavior in telecommunications, manufacturing and service operation [3], [4]. In these applications, the usefulness of PGFs is due to their capacity to capture complicated dependence while retaining analytical clarity.

The development of computational methods and numerical algorithms has expanded the analysis with the PGF further. With the combination of classical generating function methods and modern technical tools approached high-dimensional and structurally complicated systems that are previously thought intractable [5]. These advances confirm the rising importance of PGFs as a bridge between classical probabilistic theory and modern stochastic processes.

However, there are still challenges that prevent the full use of PGFs in modern applications. Many real-world systems display non-stationarity, heterogeneity, and high-dimensional interactions limiting purely analytical methods. As a result, hybrid approaches that merge PGF-based analysis with numerical inversion methods, simulation frameworks, and data-driven methods such as machine learning to extract distributions and generate stochastic trajectories have been proposed [5], [6]. The PGFs show an ability to scale to larger, more complicated stochastic problems using lots of data.

In light of these developments, the present study seeks to enhance the understanding and use of probability generating functions. To begin with, the theoretical foundations of PGFs are recapped, which include their algebraic properties and connections with discrete probability distributions. The second group of applications examines branching processes, a theory of queues, and the construction of discrete distributions for simulation and analysis. This research shows that probability generating functions (PGFs) and their use are not only theoretically rigorous but also practically significant to help solve contemporary stochastic modelling problems.

Despite the extensive historical treatment of probability generating functions (PGFs) as abstract algebraic tools, a meaningful gap remains in seamlessly integrating their theoretical rigor with contemporary, data-driven applications in stochastic modeling. This study aims to bridge that divide by revisiting the structural properties of PGFs and explicitly demonstrating their continued utility in modern distribution generation, branching processes, and queueing theory. While recent inquiries by [8] have underscored the role of PGFs in characterizing complex contagion on clustered networks, and [9] has highlighted their application in analyzing reliability within discrete stochastic systems, there is a pressing need to unify these disparate applications under a coherent analytical framework. Furthermore, current literature has expanded the use of generating functions to novel discrete distribution families for sustainable data analysis, as demonstrated by [10], and to performance metrics in manufacturing queueing systems, as explored by Saini, Singh, and [11]. However, these studies often lack a synthesized approach that connects operational insights back to fundamental extinction and stability theorems. By coupling classical Galton–Watson theory with recent advances in branching process modeling described by [12], this paper positions itself as both a theoretical consolidation and a practical guide. It illustrates how PGFs serve as essential bridges between intuitive probabilistic reasoning and rigorous computational solutions in an era of increasingly complex stochastic problems.

## 2. METHODS

Let **X** denote discrete random variable such that it takes values in the set of non-negative integers $\{0,1,2,\dots\}$, with PMF $\mathrm{P(X = k)}$ where $k \geq 0$. The PGF $G_X(t)$ of X is defined as the expectation of $t^x$, for a complex variable t, given by the mathematical expression as [7]:

$$G_X(t) = E(t^x) = \sum_{k=0}^{\infty} t^x p_k \,, \quad |t| \leq 1 \tag{1}$$

This generating function uniquely characterizes the probability distribution of X, since it encodes the whole PMF into a single analytical object, helping us find all moments and compute probabilistic quantities. The probabilities sum up to one, ensuring the radius of convergence of PGF is at least 1. This means that $G_X(t)$ is analytic on the complex unit disk. [13] The investigation of distributions and related stochastic processes can be undertaken with the help of tools from complex analysis and functional equations [14].

### 2.1 Properties of PGF

The utility of the PGF stems from several powerful properties that simplify otherwise complex operations involving probability distributions.

#### 2.1.1 Uniqueness Theorem and Probability Recovery

The PGF $G_X(t)$ defines the probability distribution of $X$ uniquely. Conversely, the PMF $\mathrm{P(X = x)}$ can be retrieved by taking derivatives of the PGF and evaluating them at $t = 0$.

**Proof**

The $G_X(t)$ is a power series. Its coefficients $\mathrm{p_k}$ are uniquely determined by Taylor series expansion around $t = 0$.

$$G_X(t) = \sum_{k=0}^{\infty} P_k t^x = P_0 + P_1 t + P_2 t^2 + P_3 t^3 + \cdots + P_k t^k + \cdots \tag{2}$$

$$At\ t = 0{:}\ G_X(0) = P_0 + 0 = P(X = 0)$$

$$G_X^I(t) = P_1 = P(X = 1) \tag{3}$$

$$G_X^k(t) = \textstyle\sum_{k=0}^{\infty} j(j-1)\dots(j-k+1)\ P_k t^{j-k}$$

$$G_X^k(0) = k(k-1)\,(1)P_k = k!\,P_k \tag{4}$$

Therefore, the probability mass function is recovered by:

$$P(X = k) = P_k = \frac{1}{k!}\,G_X^k(0) \tag{5}$$

For example, by utilizing the probability generating function (PGF) of the Poisson distribution, the corresponding probability mass function (PMF) can be derived from the PGF. The PGF of Poisson distribution of random variable X with parameter λ is given by:

$$G_X(t) = e^{\lambda(t-1)}.$$

$For\ k = 0, G_X^0(0) = \frac{1}{0!}e^{\lambda(0-1)} = e^{-\lambda}$

$For\ k = 1, G_X'(t) = \lambda e^{\lambda(t-1)}$

$$P(X = 1) = \frac{1}{1!}\lambda e^{\lambda(0-1)} = \lambda e^{-\lambda}$$

$For\ k = 2, G_X''(t) = \lambda^2 e^{\lambda(t-1)}$

$$P(X = 2) = \frac{1}{2!}\lambda^2 e^{\lambda(0-1)} = \frac{1}{2}\lambda^2 e^{-\lambda}$$

$For\ k = 3, G_X'''(t) = \lambda^3 e^{\lambda(t-1)}$

$$P(X = 3) = \frac{1}{3!}\lambda^3 e^{\lambda(0-1)} = \frac{1}{6}\lambda^3 e^{-\lambda}$$

For kth derivative, $G_X^k(t) = \lambda^k e^{\lambda(t-1)}$

$$P(X = k) = \frac{1}{k!}\lambda^k e^{\lambda(0-1)} = \frac{\lambda^k e^{-\lambda}}{k!}, k = 0, 1, 2, 3, \ldots$$

### 2.1.2 Moment Generation

The $r$-th factorial moment, $E[X(X-1)(X-2)\ldots(X-r+1)]$ can be obtained by differentiating the PGF r times and evaluating at $t = 1$.

$$E[X(X-1)(X-2)\ldots(X-r+1)] = G_X^r(1)$$

**Theorem**

The mean and variance from the moment are given by:

i. $E(X) = G_X'(1)$ and

ii. $Var(X) = G_X''(1) + G_X'(1) - \left(G_X'(1)\right)^2$

**Proof**

i. $G_X(t) = \sum_{k=0}^{\infty} P_k t^x$

$$G_X'(t) = \sum_{k=0}^{\infty} xP_k t^{x-1}$$

$$G_X'(1) = \sum_{k=0}^{\infty} xP_k = E(X)$$

ii. $G_X''(t) = \sum_{k=0}^{\infty} x(x-1)P_k t^{x-2}$

$$G_X''(t) = \sum_{k=0}^{\infty} (x^2 P_k t^{x-2} - xP_k t^{x-2})$$

$$G_X''(1) = \sum_{k=0}^{\infty} x^2 P_k - \sum_{k=0}^{\infty} xP_k$$

$$G_X''(1) = \sum_{k=0}^{\infty} x^2 P_k - G_X'(1)$$

Recall that $Var(X) = E(x^2) - (E(x))^2$ and

$E(x^2) = G_X''(1) + G_X'(1)$

$\therefore Var(X) = G_X''(1) + G_X'(1) - \left(G_X'(1)\right)^2$

### 2.1.3 The PGF of a Sum of Independent Random Variables

Let $X_1, X_2, X_3, \ldots X_n$ be mutual independent random variables. Define their sum as:

$$S_n = X_1 + X_2 + \cdots + X_n$$

Then, the PGF of $S_n$ is given by the product of individual PGFs as:

$$G_{S_n}(t) = \prod_{i=1}^{n} G_{X_i}(t)$$

where $G_X(t)$ is the common PGF of the i.i.d. random variables $X_i$. This formulation is fundamental to the analysis of branching processes and compound distributions [7][15].

**Proof**

$$\begin{aligned} G_{S_n}(t) &= E[t^{S_n}] \\ &= E[t^{X_1+X_2+\,X_3+\,\ldots+X_n}] \\ &= E[t^{X_1}.\,t^{X_2}.\,t^{X_3}\,\ldots\,t^{X_n}] \\ &= E[t^{X_1}].\,E[t^{X_2}].\,E[t^{X_3}].\,\ldots E[t^{X_n}] \\ &= G_{X_1}(t).\,G_{X_2}(t)..\,G_{X_3}(t)\,\ldots\,.\,G_{X_n}(t) \\ &= \prod_{i=1}^{n} G_{X_i}(t) \end{aligned}$$

## 2.2 PGF of Some Discrete Probability Distributions

**Table 1:** PGF of Some Discrete Probability Distributions

| S/N | Probability Distribution | PMF | PGF |
|---|---|---|---|
| 1 | Bernoulli | $p^x q^{1-x}, x = 0,1$ | $pt + q, \quad for\ q = 1 - p$ |
| 2 | Binomial | $\binom{n}{x} p^x q^{n-x}, x = 0,1, \ldots n$ | $(pt + q)^{\boldsymbol{n}}$ |
| 3 | Poisson | $\frac{\lambda^x e^{-\lambda}}{x!}, x = 0,1,2..$ | $e^{\lambda(t-1)}$ |
| 5 | Geometric (Type I - # trials to 1st success) | $pq^{x-1}, x = 1, 2, 3..$ | $\frac{p}{1 - qt}, \quad \lvert t\rvert < \frac{1}{1-p}$ |
| 6 | Geometric (Type II - # failures before 1st success) | $pq^x, x = 0,1, 2, ..$ | $\frac{pt}{1 - qt}, \lvert t\rvert < \frac{1}{1-p}$ |
| 7 | Negative Binomial | $\binom{x + r - 1}{r - 1} p^r q^x, x = 0,1,2 \ldots$ | $\left(\frac{pt}{1 - qt}\right)^r, \lvert t\rvert < \frac{1}{1-p}$ |
| 8 | Multinomial | $\frac{n!}{x_1!\, x_2! \ldots x_k!} p_1^{x_1} p_2^{x_2} \ldots p_k^{x_k}$ | $(p_1 t_1 + p_2 t_2 + \cdots + \ p_k k)^{\boldsymbol{n}}$ |
| 9 | Discrete Uniform | $\frac{1}{n!}, x = 0,1,2 \ldots, n$ | $\frac{t(1 - t^n)}{n(1 - t)}, t \neq 1$ |

## 2.3 Applications of PGF

Three applications of PGF are discussed as follow:

### 2.3.1 Derivation of Probability Distribution.

Using the PGF of Geometric distribution, the number of failures before the first success is given by

$$\mathrm{G_X(t)} = \frac{pt}{1 - qt}$$

Let $X \sim Negative\ Binomial(r, p)$ be the number of failures before the rth success. Then, $X$ is the sum of $r$ independent geometric distribution random variables given as:

$$X = \ Y_1 + \ Y_2 + Y_3 + \cdots + \ Y_r$$

such that $Y_i \sim \ Geometric\ (p)$. Then the PGF of X is the product of the PGFs of $Y_i$

$$\begin{aligned} \mathrm{G_X(t)} &= \ \mathrm{G}_{(Y_1+\,Y_2+Y_3+\cdots+\,Y_r)}(\mathrm{t}) \\ &= \ \mathrm{G}_{Y_1}(\mathrm{t}).\,\mathrm{G}_{Y_2}(\mathrm{t}) \ldots \mathrm{G}_{Y_r}(\mathrm{t}) \\ &= [\mathrm{G}_Y(\mathrm{t})]^r \end{aligned} \tag{6}$$

$$\mathrm{G_X(t)} = \left(\frac{pt}{1 - qt}\right)^{\boldsymbol{r}}, \lvert t\rvert < \frac{1}{1-p}$$

We proceed by expanding the PGF above to obtain the PMF of Negative Binomial distribution. Using binomial series expansion:

$$(1 - k)^{\boldsymbol{r}} = \binom{x + r - 1}{r - 1} k^r, \lvert k\rvert < 1 \tag{7}$$

Putting $k = (1 - qt)$, we have

$$G_X(t) = \binom{x+r-1}{r-1} p^r q^x\, t^x \tag{8}$$

Note, by definition of PGF, we have:

$$G_X(t) = \sum_{k=0}^{\infty} P_k t^x \tag{9}$$

Comparing (4.8) and (4.9), the PMF of Negative Binomial is obtained as:

$$P(X = x) = \binom{x+r-1}{r-1} p^r q^x, x = 0,1,2,\ldots \tag{10}$$

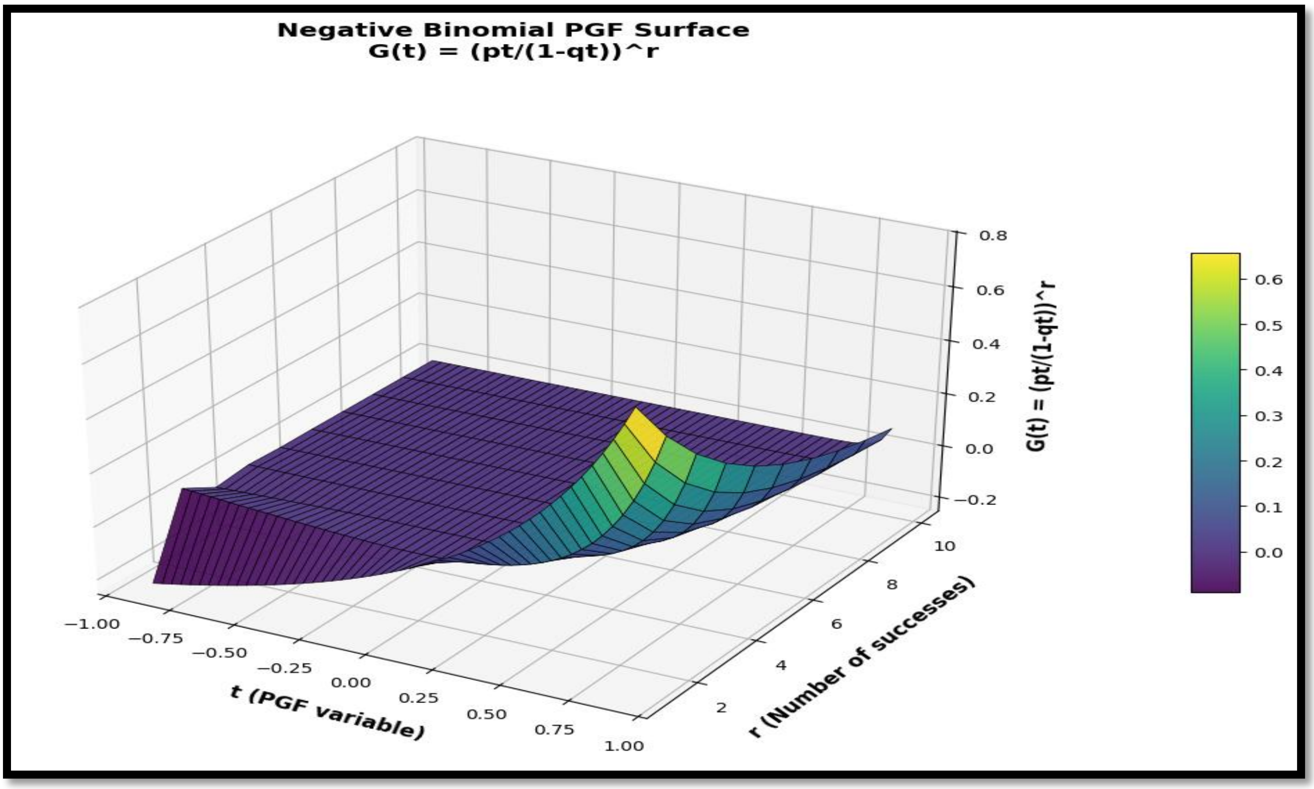


**Figure 1:** Plot of Negative Binomial PGF surface

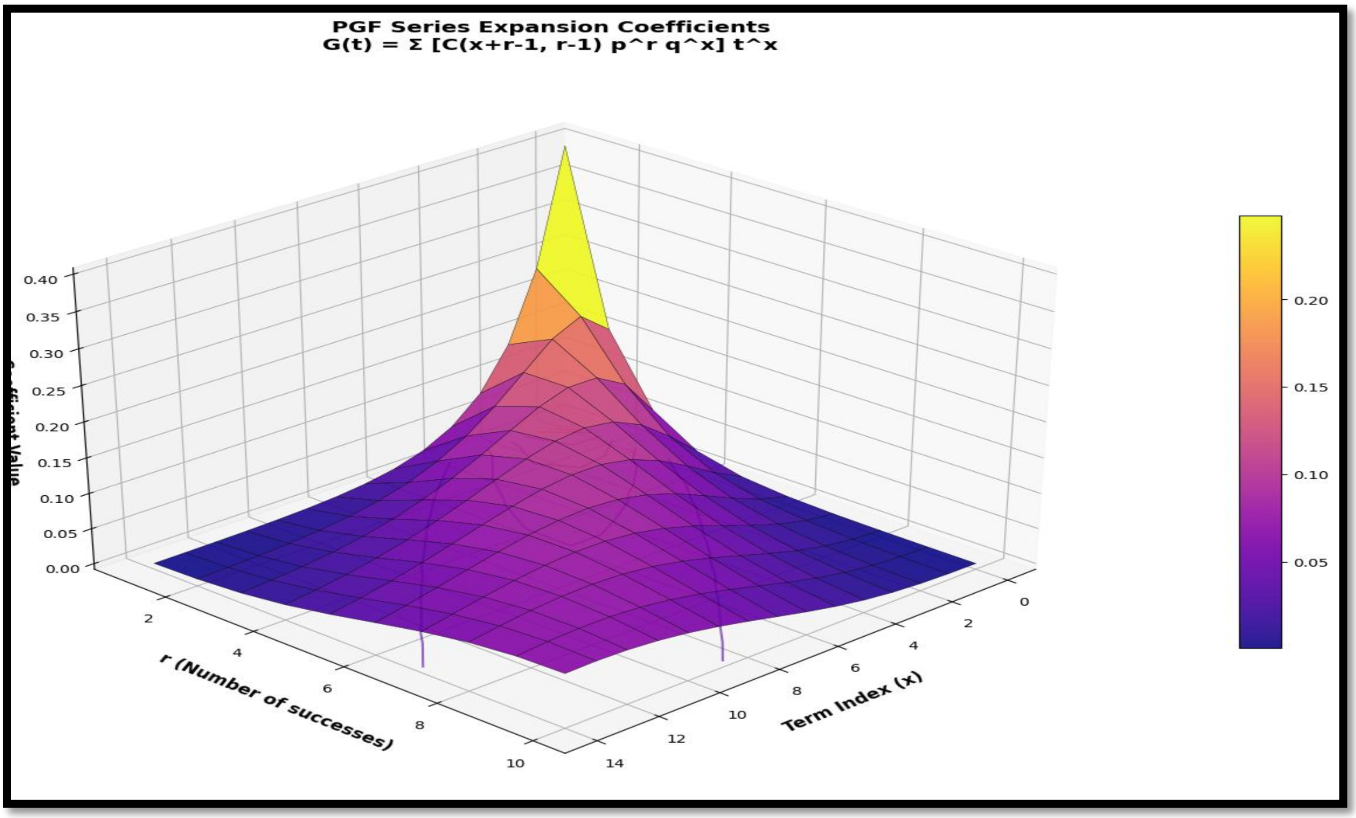


**Figure 2**: Plot of PGF series Expansion coefficient

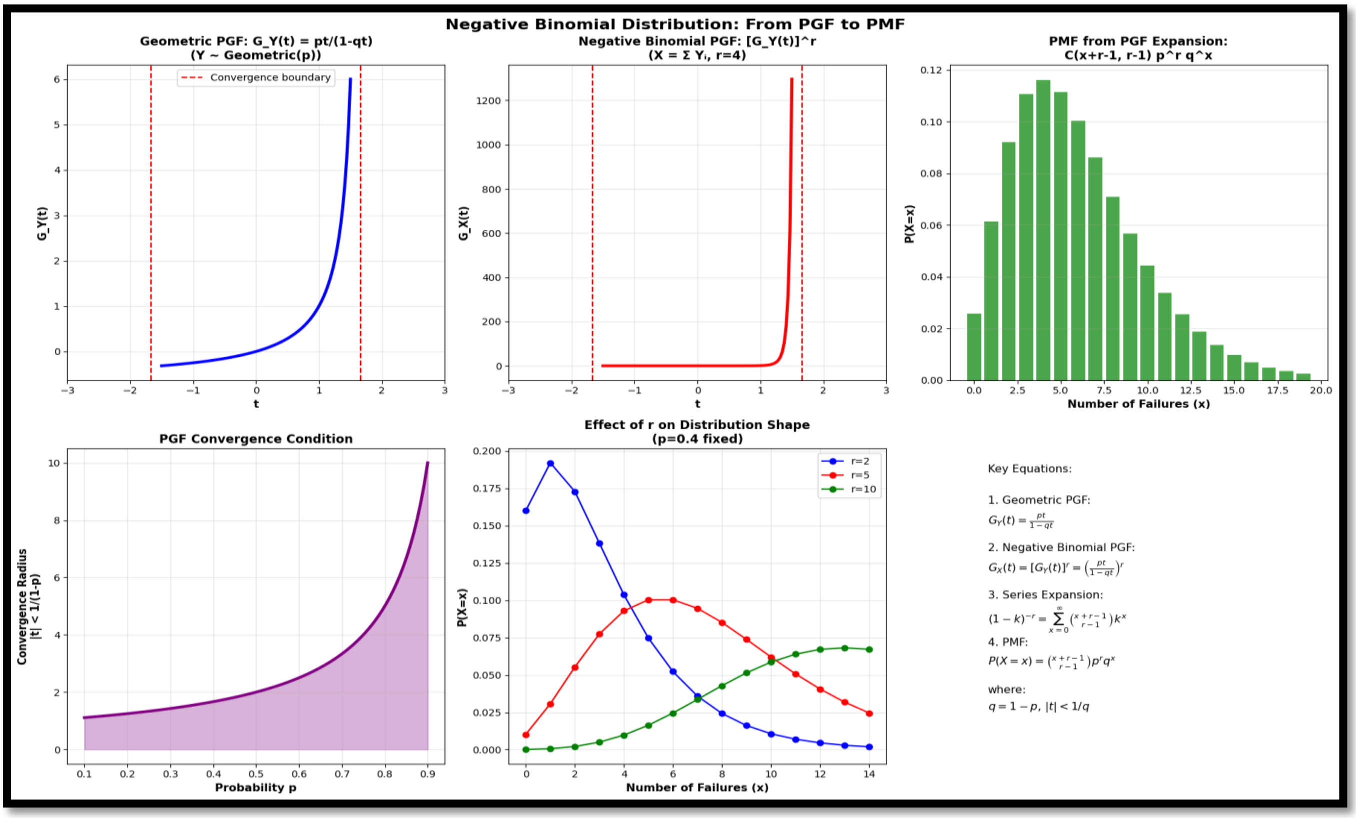


**Figure 3**: Negative Binomial Derivation from Geometric PGF Framework

### 2.3.2 Application of PGF to Branching Process

The probability generating function (PGF) offers a compact and efficient approach for examining the generational dynamics and long-term behavior of discrete-time branching processes. A branching process

models a population in which each individual in generation $n$ independently produces a random number of offspring according to a fixed probability distribution. Let $Z_n = population\ size\ in\ generation\ n$ and X = no of offspring produced by single individual. $p_k = P(X = k), k = 0,1,2, \ldots$
The PGF of the offspring random variable X is defined as:

$$G_X(t) = \sum_{k=0}^{\infty} p_k t^k \tag{11}$$

Each individual in generation n reproduces independently according to the offspring distribution X. Hence, the population size in generation n+1 is the sum of $Z_n$ independent copies of X, given by:

$$Z_n = X_1 + X_2 + X_3 + \cdots + X_{Z_n}$$

$$G_{Z_{n+1}}(t/Z_n = k) = [G_X(t)]^k \tag{12}$$

Since $Z_n$ is random, the expectation over $Z_n$ gives:

$$\begin{aligned} G_{Z_{n+1}}(t) &= E[[G_X(t)]^{Z_n}] \\ &= G_{Z_n}(G_X(t)) \end{aligned} \tag{13}$$

Thus, the recursive formula for the population PGF.
Let us assume the process starts with one individual in generation zero, that is $Z_0 = 1$, then the PDF of $Z_0$ is given by:

$$G_{Z_0}(t) = t$$

Using the recursive relationship,
$G_{Z_1}(t) = G_X(t)$
$G_{Z_2}(t) = G_X(G_X(t))$
$G_{Z_3}(t) = G_X\big(G_X(G_X(t))\big)$
and, in general, we have

$$G_{Z_n}(t) = G_X\big(G_X \ldots (G_X(t) \ldots)\big) \tag{14}$$

**The Probability of Extinction**

The probability of extinction in a branching process refers to the likelihood that the population size eventually declines to zero, resulting in the complete disappearance of the population over time. Let

$$q_n = P(Z_n = 0),$$

the probability that the population has become extinct by generation n. By definition of the PGF,

$$q_n = G_{Z_n}(0)$$

$$q_{n+1} = G_X(q_n)$$

and as $n \to \infty$, the sequence $\{q_n\}$ converges to the ultimate extinction probability

$$q = \lim_{n\to\infty} q_n$$

This satisfies the fixed-point equation

$$q = G_X(q)$$

where $q$ is the smallest solution in the interval [0,1]. Mathematically, if the mean offspring $\mu = G_X'(1) \leq 1$, extinction is certain ($q = 1$); if $\mu > 1$, the extinction probability satisfies $q < 1$.

Figure 4 illustrate the discrete-time evolution probability $Q_n$ across generation n for three branching processes with distinct mean offspring numbers $\mu$. The supercritical case $\mu = 1.3$, $Q_n$ rapidly converges to a stable extinction probability $q_\infty = 0.4 < 1$, which reflect a positive survival probability. The critical ($\mu = 1$) and subcritical ($\mu = 0.5$) regimes, $Q_n$ approaches 1, confirming certain extinction ($q_\infty = 1$), with convergence marked slower at the critical threshold.

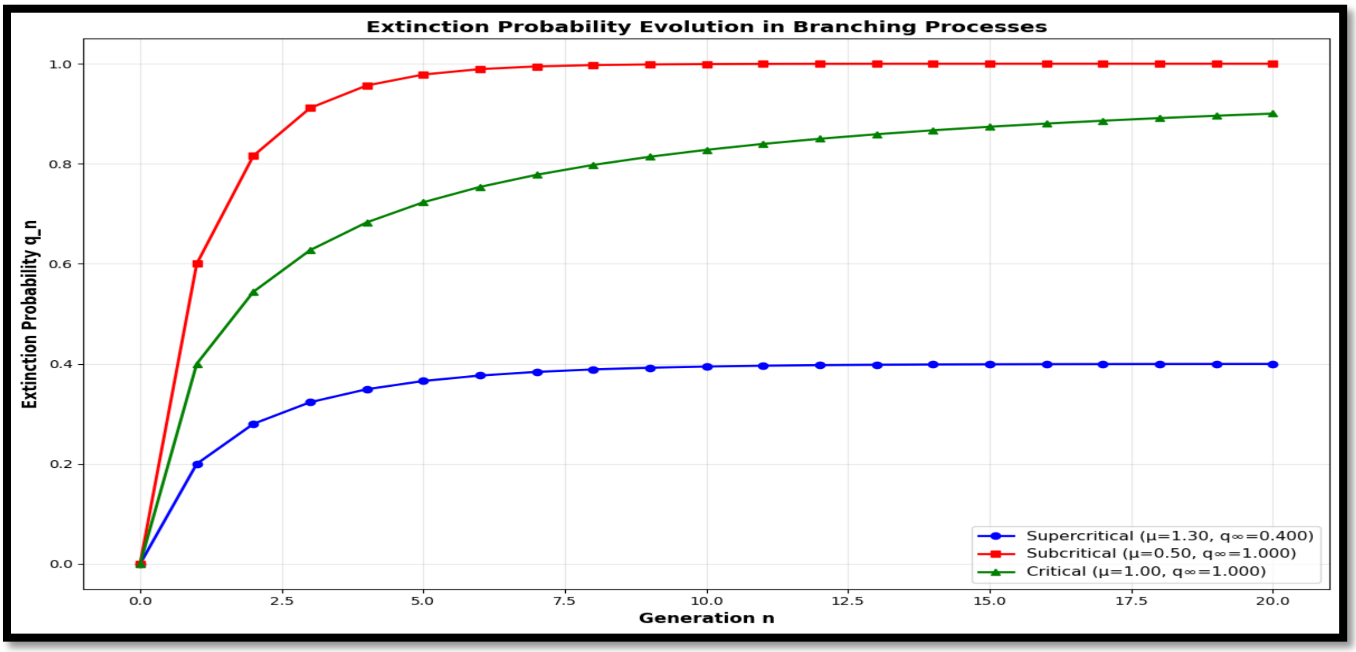


**Figure 4**: Extinction Probability Evolution in Branching Processes

### 2.4 Application of Probability Generating Function to Queuing Theory

This section presents the use of the probability generating function (PGF) as an analytical tool for studying queue-length behavior in an $M/M/1$ queuing system, highlighting its effectiveness in obtaining steady-state characteristics in a compact form. The system is defined by a Poisson arrival process with rate $\lambda$ and exponentially distributed service times with rate $\mu$, operating with a single server. Under steady-state conditions, the random variable $N$ represents the number of customers in the system, and the PGF-based formulation facilitates an efficient derivation of the corresponding probability distribution as well as key performance measures.

If we let $P(N = n)$ denote the steady-state probability that the system contains n customers. Under the stability condition, the probability is given as:

$$P(N = n) = (1 - \rho)\rho^n, n = 0,1,2, \dots$$

where

$$\rho = \frac{\lambda}{\mu} < 1$$

is the intensity.

The PGF for the steady-state queue length N is expressed as

$$G_N(t) = E[t^N] = \sum_{n=0}^{\infty} P(N = n)t^n$$

$$= (1 - \rho)\sum_{n=0}^{\infty}(\rho t)^n$$

$$= \frac{1 - \rho}{1 - \rho t}$$

Using the properties of PGF in derivation of mean and variance, then the expected number of customers in the system is obtained as [16]:

$$E[N] = \frac{\rho}{1 - \rho}$$

and the variance of the queue length is obtained as:

$$Var[N] = \frac{\rho}{(1 - \rho)^2}$$

Hence,

(a) The steady-state probability at $t = 0$ is obtained as:

$$P(N = n) = \frac{1}{n!}\frac{d^n}{dt^n}G_N(t) = (1-\rho)\rho^n$$

(b) The expected waiting time using Littles law is obtained as:

$$E[W] = \frac{E[N]}{\lambda} = \frac{\rho}{\lambda(1-\rho)}$$

(c) The probability of Congestion, for a given threshold k is obtained as:

$$P(N > k) = 1 - \sum_{n=0}^{k} P(N = n)$$

### 2.4.1 Simulation on M/M/1 Queue PGF Properties

This study presents a simulation-validated probability generating function analysis of an M/M/1 queue to evaluate steady-state behavior, performance measures, and congestion characteristics under moderate traffic intensity.

**(a) Model Description and Parameters**

The system under study is a single-server Markovian queue (M/M/1) with Poisson arrivals and exponential service times.

| **Parameter** | **Description** | **Value** |
|---|---|---|
| Λ | Arrival rate | 0.75 |
| M | Service rate | 1.00 |
| P=λ/μ | Traffic intensity | 0.75 |

**Note:** Since $\rho < 1$, the system is stable and admits a steady-state solution

(b) **Probability Generating Function (PGF) Evaluation**

The probability generating function of the steady-state queue length N is computed at specific points of interest.

| **Z** | $G_N(z)$ |
|---|---|
| 0.0 | 0.2500 |
| 0.5 | 0.4000 |
| 1.0 | 1.0000 |

**(c) Performance Metrics**

Key system performance indicators are summarized below.

| **Metric** | **Symbol** | **Value** |
|---|---|---|
| Expected queue length | E[N] | 3.0000 |
| Queue length variance | Var[N] | 12.0000 |
| Expected waiting time | E[W] | 4.0000 |

**d. Congestion Analysis (Tail Probabilities)**

The probability of excessive congestion is evaluated using the geometric steady-state distribution.

| **Threshold** | **Probability** |
|---|---|
| P(N>5) | 0.177979 |
| P(N>10) | 0.042235 |
| P(N>15) | 0.010023 |

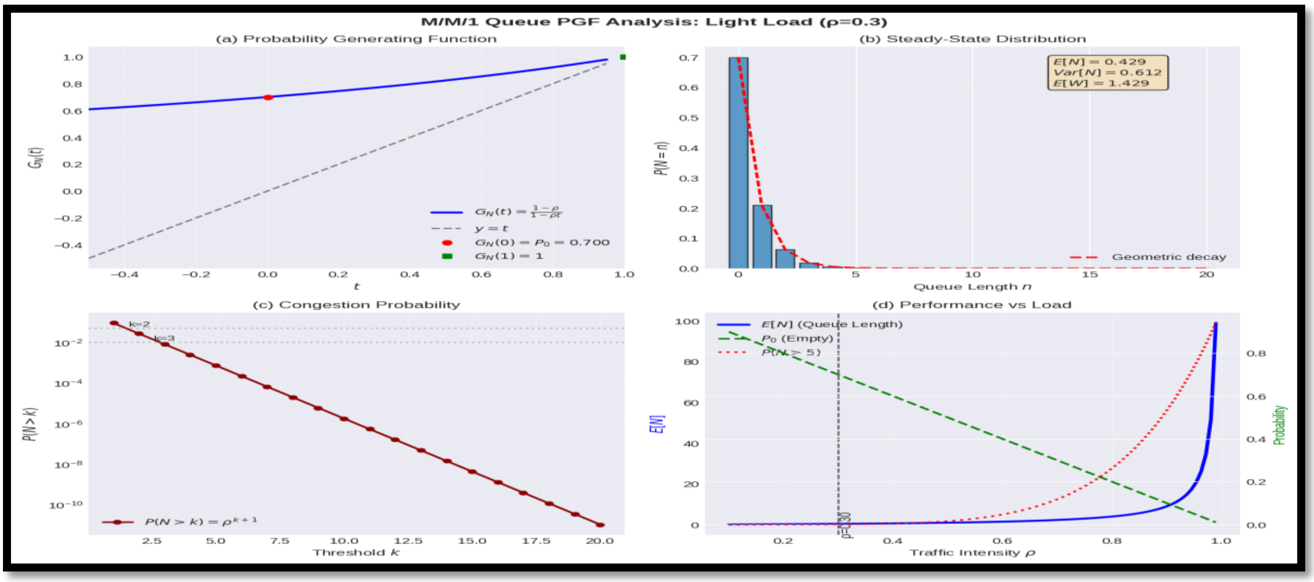


**Figure 5**:M/M/1 Queue PGF Analysis for Light Load (0.3)

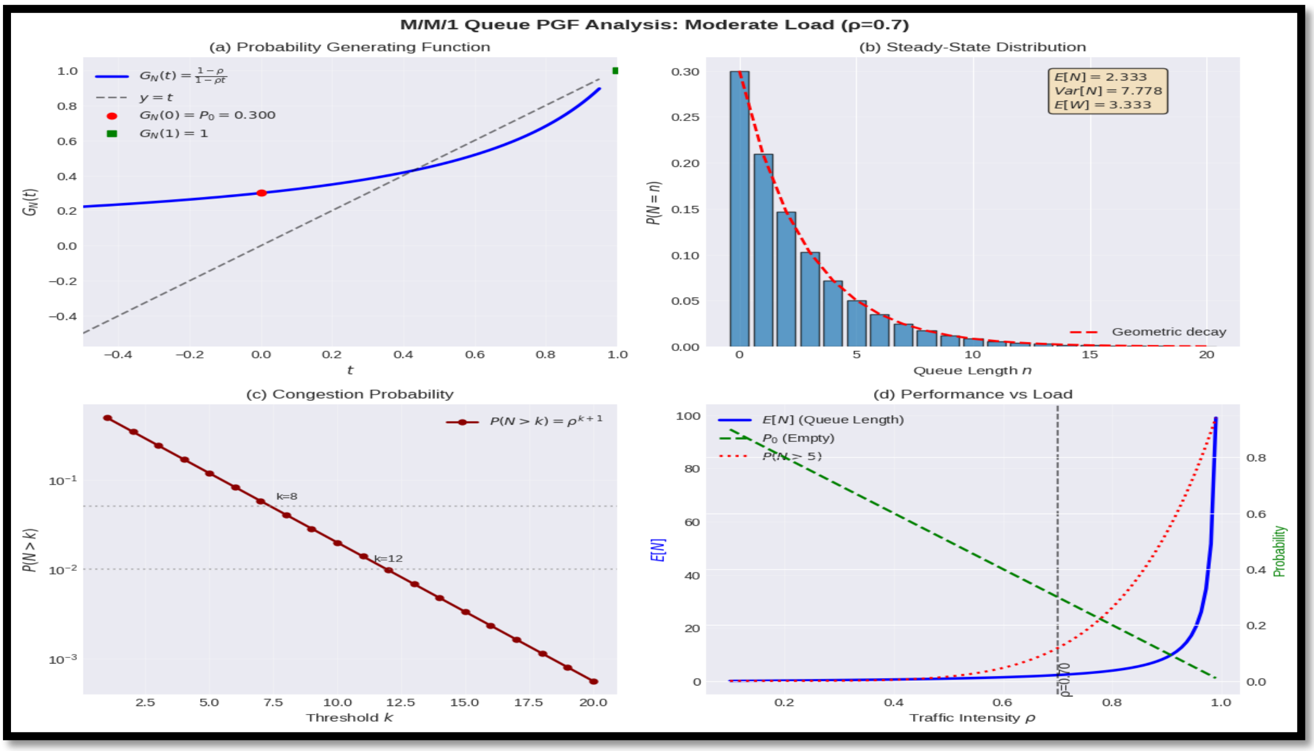


**Figure 6**: M/M/1 Queue PGF Analysis for Moderate Load (0.7)

Figure 5 and 6 illustrates the PGF $G_0(t) = 0.6t^{0.6}$. The stable output for $t \in [0,1]$ versus erratic values for $t < 0$ demonstrate the essential domain restriction for valid PGFs of non-negative integer random variable.

## 3. DISCUSSION

The results of this study are consistent with the classical foundations of probability generating functions (PGFs) while also reinforcing their continued relevance in modern discrete stochastic modelling. Early theoretical work established PGFs as a central tool for characterizing discrete random variables, particularly due to their ability to uniquely encode probability mass functions, generate moments, and simplify convolution [7]. The present findings reaffirm these foundational properties and demonstrate how they remain analytically powerful across a range of stochastic applications.

Compared with traditional textbook treatments, which often emphasize the abstract algebraic structure of PGFs [7], this study adopts a more application-oriented perspective. By explicitly linking PGF properties to distribution generation, branching processes, and queueing systems, the analysis highlights how PGFs function not only as theoretical constructs but also as practical modelling instruments. The derivation of the negative binomial distribution from geometric components, for example, reflects the classical compounding interpretation of PGFs, while also offering a clear probabilistic narrative for waiting-time phenomena and aggregation effects.

In the context of branching processes, the results closely align with established Galton–Watson theory, particularly the characterization of extinction probability as the smallest fixed point of the offspring PGF. This condition, which relates extinction behavior to the mean number of offspring, has been rigorously analyzed in both classical and applied studies [7],[3]. The present study confirms these results and emphasizes the interpretive strength of the PGF framework in connecting individual-level reproduction mechanisms with long-term population outcomes. Compared to earlier works that focus on asymptotic lifetime distributions or power-law behavior [3], the approach adopted here prioritizes analytical transparency and computational accessibility.

The queueing theory application further situates the findings within a well-established body of literature on Markovian service systems. The steady-state results obtained for the M/M/1 queue are consistent with classical queueing theory, where geometric queue-length distributions and associated performance measures are well documented [1]. However, while traditional analyses often rely on balance equations or embedded Markov chain arguments, the PGF-based formulation provides a compact and unified framework for deriving both moments and tail probabilities. This is particularly valuable in congestion analysis, where tail behavior conveys operational risks not captured by mean performance measures alone.

Moreover, the simulation-supported analysis demonstrates that PGFs remain effective even when combined with numerical evaluation, reflecting broader methodological trends in stochastic modelling. Recent research has emphasized the importance of integrating classical probabilistic tools with computational techniques to address increasingly complex systems [5]. Within this context, PGFs offer a mathematically rigorous backbone that supports simulation-based validation while retaining interpretability and analytical coherence.

The theoretical properties of PGFs are well established in the literature, this study contributes by synthesizing classical results with application-driven analysis and simulation evidence. In contrast to earlier works that treat PGFs primarily as abstract analytical devices, the present discussion demonstrates their enduring practical relevance in discrete stochastic modelling, particularly for branching phenomena and queueing systems encountered in contemporary applications.

## 4. CONCLUSION AND SUGGESTION

This study establishes that probability generating functions (PGFs) continue to serve as a fundamental and highly effective framework for the analysis of discrete stochastic phenomena. By synthesizing their theoretical properties with applications to distribution construction, branching processes, and queueing models, the work demonstrates that PGFs offer both analytical rigor and interpretive clarity. The results indicate that a range of key probabilistic quantities, including moments, extinction probabilities, and steady-state performance measures, can be derived in a coherent and transparent manner. Furthermore, the close correspondence between analytical predictions and simulation outcomes underscores the robustness and practical reliability of the PGF framework in characterizing the behavior of discrete stochastic systems.

Future research directions suggest considerable potential to extend the applicability of PGFs to more complex and realistic modeling scenarios. Investigations may focus on systems exhibiting dependence, non-stationarity, or higher-dimensional state spaces, where purely analytical solutions are often infeasible. The integration of PGFs with numerical techniques, simulation, or data-driven approaches provides a pathway to retain the theoretical strengths of classical probability while effectively addressing the challenges posed by modern stochastic systems. These developments are expected to reinforce the relevance of PGFs as a versatile and insightful tool for both theoretical research and applied stochastic modeling

**Conflict of interest**: The authors declare that none of them have any conflicting interests.

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