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# A Review of Convolution of Geometric Distributions and Its Properties

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**Abstract**

This review explores the convolution of geometric distributions, a key operation in probability theory for deriving the distribution of the sum of independent random variables. Geometric distributions quantify the number of Bernoulli trials needed for the first success and are foundational in discrete probability models. Convolving multiple geometric distributions with a common success probability produces a negative binomial distribution, modelling the number of trials needed to achieve a given number of successes. We present a concise derivation of this result, highlighting the relationship between geometric and negative binomial distributions. The review also outlines essential properties of the negative binomial distribution, including its mean, variance, moment-generating function, and some applications.

## 1. INTRODUCTION

The geometric distribution is a fundamental discrete probability distribution commonly employed in modeling the number of trials until the first success in a sequence of independent Bernoulli trials. Due to its simplicity and practical relevance in fields such as reliability engineering, survival analysis, and queuing theory, the study of its convolution properties, whereby sums of independent geometric random variables are considered, has garnered significant attention in probability theory and applied statistics. Convolution of geometric distributions plays a pivotal role in understanding compound distributions, renewal theory, and count data modeling, providing critical insight into the behavior and properties of sums of discrete random variables [1]. Moreover, convolution techniques have applications in areas such as insurance mathematics, stochastic processes, and operational risk management, underscoring their theoretical and practical importance [4],[5].

Classically, the convolution of geometric distributions has been approached through the use of generating functions and recursive algorithms, which facilitate the derivation of distributional properties and exact expressions for the sum of independent geometric variables. Recent advancements have highlighted the characterization of extended geometric distributions and their mixtures through convolution equations and generating functions, which further generalize the classical geometric distribution and link it to related distributions like the negative binomial and gamma distributions [2], providing a broader framework for analytical and computational studies. The analysis of such convolutions not only deepens theoretical understanding but also promotes efficient computation strategies essential for practical applications, as demonstrated by alternative methods dealing with compound negative binomial and gamma distributions [2].

The significance of convolution methods extends to modern applied problems, where the aggregation of discrete variables modeled by geometric distributions emerges naturally. For instance, the summation of Poisson-distributed random variables, a problem closely related to geometric distribution convolutions, has been studied using convolution method approaches, which offer promising advantages in handling sums of discrete random variables with applications to statistical inference and environmental modeling [3]. These convolution-based approaches provide a unified and computationally tractable technique, instrumental for probabilistic modeling in areas where discrete event counts and arrivals are key phenomena. Such contributions highlight the ongoing evolution of convolution theory in discrete distributions and its crucial role in advancing applied statistical methodologies.

In addition to theoretical developments, convolution operations of geometric distributions have found relevance in broader multidisciplinary domains involving stochastic modeling of count data and renewal processes. Renewal theory extensively uses the convolution of waiting time distributions, with geometric distributions being a key example, to model the occurrence of random events over time. This is exemplified in discrete Markov chains and renewal reward processes, where convolution-based Laplace-Stieltjes transform methods are applied for performance evaluation and optimal policy derivation [6]. Likewise, advancements in probabilistic recursions and efficient computation techniques for compound distributions have benefitted from convolution formulations, enabling their use in actuarial science, reliability analysis, and operational risk capital calculations [7][4].

Despite its long-standing history, the convolution of geometric distributions remains an active area of research due to its intrinsic complexity and versatile applicability. Recent studies underscore the need for continued investigation into exact and approximate convolution formulas, computational algorithms, and the characterization of convoluted distributions to enhance their usability in real-world problems. Furthermore, exploring convolution in extended distributional frameworks, such as those involving the Harris or Mittag-Leffler-type geometric distributions, promises to expand theoretical understanding and applicability in analytic function spaces and fractional calculus domains [7][8]. This review synthesizes existing literature on the convolution of geometric distributions, highlights methodologies, properties, and applications, and provides a comprehensive assessment of recent advancements, including the seminal work on sums of Poisson-distributed variables using convolution methods [ 3].

Probability generating functions constitute a central analytical instrument in the study of discrete random variables, particularly in contexts involving sums of independent components and convolution structures, as emphasized by [10]. In this framework, the Negative Binomial distribution naturally emerges as the distribution of a sum of independent Geometric random variables, a characterization that has been systematically investigated to obtain distributional bounds and assess approximation accuracy, notably in the work of [12]. Subsequent contributions, including those of [11], have further demonstrated the adaptability of generating function techniques in extending classical results to more intricate distributional forms. Anchored in these foundational and contemporary developments, and consistent with recent theoretical refinements reported [9], the present study systematically applies the PGF framework to reconstruct the Negative Binomial distribution from its Geometric basis, offering a coherent and methodologically transparent account of their probabilistic interconnectedness.

## 2. METHODS

Let X and Y be two independent discrete random variables with probability mass functions (PMFs) $P_X(x)$ and $P_Y(y)$, defined over their respective supports $R_X$ and $R_Y$. The sum Z = X + Y is a discrete random variable and its PMF, $P_Z(z)$, is given by the convolution of $P_X(x)$ and $P_Y(y)$:

$$P_Z(z) = \sum_{x \in R_X} P_X(x) P_Y(z - x) \tag{1}$$

where the summation is taken over all x in the support of X such that $P_Y(z-x)$is defined $\left(z - x \in R_y\right)$. [13],[14]
Let $G_i \sim Geometric(p), i = 1, 2, \ldots, r,$ where $G_i$ denotes the number of failures before the i-th success in a sequence of Bernoulli trials with success probability $p \in (0,1)$. Let random variable X be defined as:

$X = \sum_{i=1}^{r} G_i$

The PMF of each Geometric random variable is:

$P(G = k) = p(1-p)^k, \quad k = 0, 1, 2, 3, \ldots \qquad (2)$

Let $G_{1,}G_2 \sim Geometric(p)$, independent. The sum $Z = G_1 + G_2$ has a distribution obtained by convolution:

$$\begin{aligned} P_Z(z) &= \sum_{i=1}^{k} P(G_1 = i).P(G_2 = k-i) \\ &= \sum_{i=1}^{k} (1-p)^i p\,.(1-p)^{k-i} p \\ &= (k+1)P^2(1-p)^k \end{aligned} \qquad (3)$$

This is the PMF of the sum of two independent Geometric random variables.

This is a Negative Binomial with parameters $r = 2$ , which models the number of failures before achieving 2 successes.

Using recursive convolution procedure, let $f_r(k) = P(X = k)$, the PMF of the sum of $r \;\; i.i.d \; Geometric(p)$ random variables.

For base case $(r \; = \; 1)$

$$f_1(k) = \; p(1-p)^k$$

For $(r \; = \; 2)$

$$f_1(k) = \; (k+1)P^2(1-p)^k$$

And for r Geometric random variable,

$$f_r(k) = \binom{k+r-1}{k} P^r(1-p)^k \qquad (4)$$

Now $f_{r+1}(k)$ using convolution:

$$f_{r+1}(k) = \sum_{j=0}^{k} \binom{j+r-1}{j} P^r(1-p)^k.p(1-p)^{k-j} \qquad (5)$$

$$\begin{aligned} &= P^{r+1}(1-p)^k \sum_{j=0}^{k} \binom{j+r-1}{j} \\ &= P^{r+1}(1-p)^k \binom{k+r}{k} \end{aligned} \qquad (6)$$

Since from Binomial Expansion, the identity:

$$\sum_{j=0}^{k} \binom{j+r-1}{j} = \binom{k+r}{k}$$

This therefore validate that:

$$P(X = k) = \binom{k+r-1}{k} P^r(1-p)^k, k = 0,1,2, \ldots \qquad (7)$$

This is PMF of Negative Binomial $NB(r, p)$

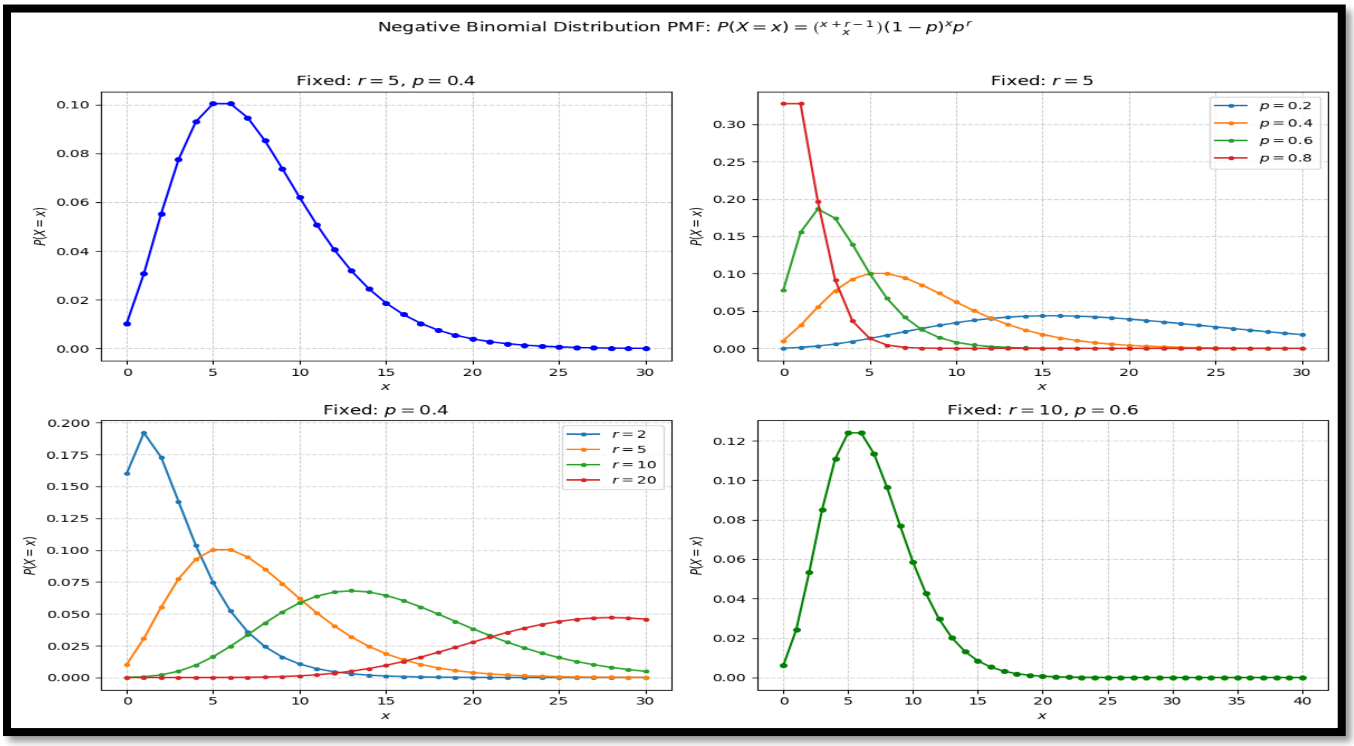


**Figure 1:** *Plot of the Pmf*

## Cumulative Distribution Function (CDF)

The CDF was derived as:

$$F(k)=\sum_{j=0}^{k}\binom{j+r-1}{j}(1-p)^{j}P^{r} \tag{8}$$

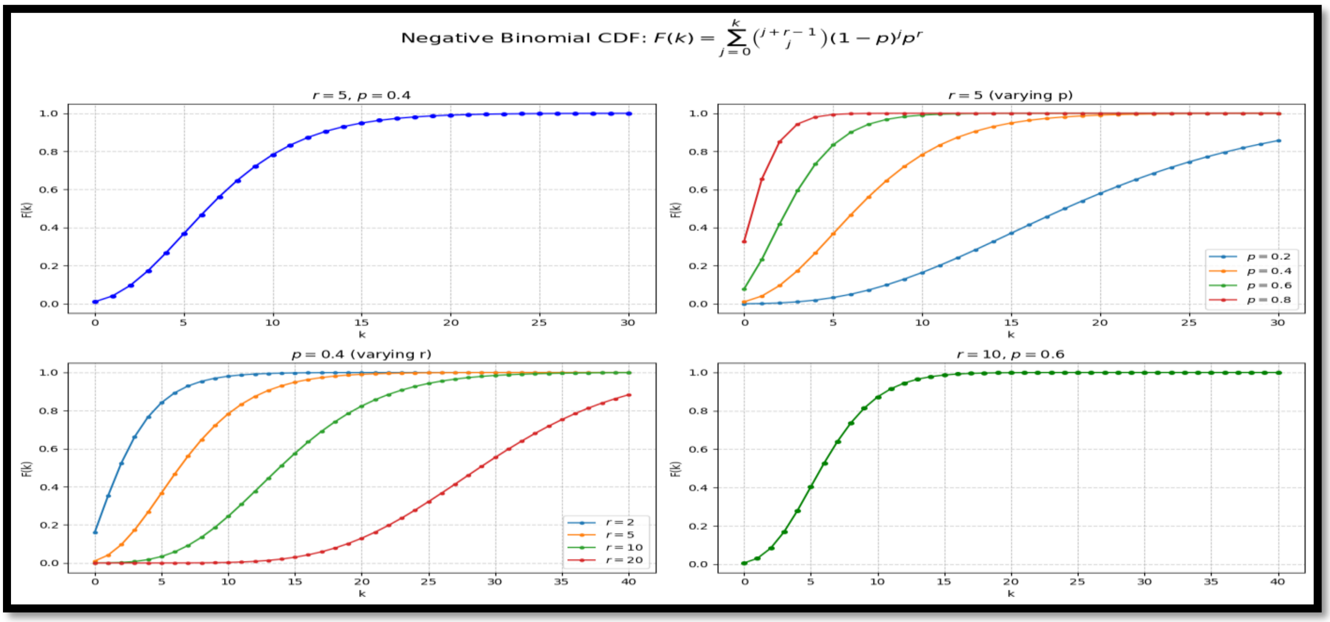


**Figure 2**: *Plot of the CDF*

## 3. RESULTS AND DISCUSSION

This section presents the key findings of the study through a systematic examination of the statistical and mathematical properties of the negative binomial distribution. The analysis begins with parameter estimation, followed by the derivation of the moment generating function, probability generating function, and cumulant generating function. Attention is then given to the survival and hazard functions, supported by graphical illustrations to aid interpretation. The section further discusses the asymptotic properties of the distribution and their implications for large-sample behavior. Finally, important distributional characteristics, including the mean, variance, skewness, kurtosis, and mode, are examined, alongside a discussion of the practical applications of the negative binomial distribution in modeling count data and overdispersion in real-world contexts.

### 3.1 Parameter Estimation

Given random variables $x_1, x_2, \ . \ . \ . \ , x_{n,}$, the log-likelihood function $\mathrm{L}(r,p)$ is :

$$\log\mathrm{L}(\mathrm{r},\mathrm{p}) = \sum_{\mathrm{i}=1}^{\mathrm{n}}\left[\log\binom{\mathrm{x_i}+\mathrm{r}-1}{\mathrm{x_i}} + \mathrm{r}\log\mathrm{p} + \ \mathrm{x_i}\log\left(1-\mathrm{p}\right)\right]$$

$$= \sum_{\mathrm{i}=1}^{\mathrm{n}}\left[\log\binom{\mathrm{x_i}+\mathrm{r}-1}{\mathrm{x_i}} + \mathrm{r}\log\mathrm{p} + \ \mathrm{x_i}\log\left(1-\mathrm{p}\right)\right]$$

$$= \sum_{\mathrm{i}=1}^{\mathrm{n}}[\log\Gamma(\mathrm{x_i}+\mathrm{r}) - \log\Gamma(r) - \log\Gamma(\mathrm{x_i}) + \mathrm{r}\log\mathrm{p} + \ \mathrm{x_i}\log\left(1-\mathrm{p}\right)]$$

$$\frac{\partial logL}{\partial p} = \sum_{i=1}^{n}\left[\frac{r}{p} - \frac{x_i}{1-p}\right] = \frac{nr}{p} - \frac{\sum x_i}{1-p} = 0$$

Solving for p,

$$\hat{p} = \frac{r}{r+\bar{x}} \tag{9}$$

r can be solved numerically using Newton-Raphson method.

**Theorem**

The Moment Generating Function MGF, Probability Generating Function PGF and the Cumulants Function of the Negative Binomial distribution are:

*i.*
$$M_x(t) = \left(\frac{p}{1-(1-p)e^t}\right)^r \tag{10}$$

*ii.*
$$G_x(t) = \left(\frac{p}{1-(1-p)t}\right)^r \tag{11}$$

*iii.*
$$K_x(t) = -\mathrm{r}\ln(1-(1-\mathrm{p})\mathrm{e}^{\mathrm{t}}) \tag{12}$$

**Proof**

i. The MGF $M_x(t)$ is defined as:

$$\mathrm{M_x(t)} = \sum_{\mathrm{x}=0}^{\infty}\mathrm{e^{tx}}\binom{\mathrm{k}+\mathrm{r}-1}{\mathrm{x}}\mathrm{P^r}(1-\mathrm{p})^{\mathrm{x}}$$

$$= \ \mathrm{p^r}\sum_{\mathrm{x}=0}^{\infty}\binom{\mathrm{k}+\mathrm{r}-1}{\mathrm{x}}[(1-\mathrm{p})\mathrm{e^t}]^{\mathrm{x}}$$

$$= p^r[1-(1-p)e^t]^{-r}$$

$$M_x(t) = \left(\frac{p}{1-(1-p)e^t}\right)^r \text{ proved}$$

ii. The PGF $G_x(t)$ is defined as:

$$G_x(t) = \sum_{x=0}^{\infty} t^x \binom{k+r-1}{x} P^r (1-p)^x$$

$$G_x(t) = p^r \sum_{x=0}^{\infty} \binom{k+r-1}{x} [(1-p)t]^x$$

$$G_x(t) = p^r[1-(1-p)t]^{-r}$$

$$G_x(t) = \left(\frac{p}{1-(1-p)t}\right)^r \text{ proved}$$

iii. The Cumulants Function $K_x(t)$ is defined as:

$$K_x(t) = \ln M_x(t) = r\ln\left(\frac{p}{1-(1-p)e^t}\right)$$

$$K_x(t) = r[lnp - ln(1-(1-p)e^t)]$$

$$K_x(t) = -r\ln(1-(1-p)e^t)$$

### 3.2 The Survival and Hazard function of Negative Binomial (NB)

The survival function S(x) and Hazard function h(x) of NB are given as:

$$S(x) = P(X > x) = 1 - F(x)$$

$$S(x) = \sum_{K=x+1}^{\infty} \binom{k+r-1}{k} P^r (1-p)^k \tag{13}$$

This expression gives the tail probability, i.e., the probability that more than x failures occur before achieving r successes.

$$h(x) = \frac{f(x)}{S(x)} = \frac{\binom{k+r-1}{x} P^r (1-p)^x}{\sum_{K=x+1}^{\infty}\binom{k+r-1}{k} P^r (1-p)^k} \tag{14}$$

This is the instantaneous risk of observing the x-th failure before the r-th success, given it hasn't occurred in fewer than x failures.

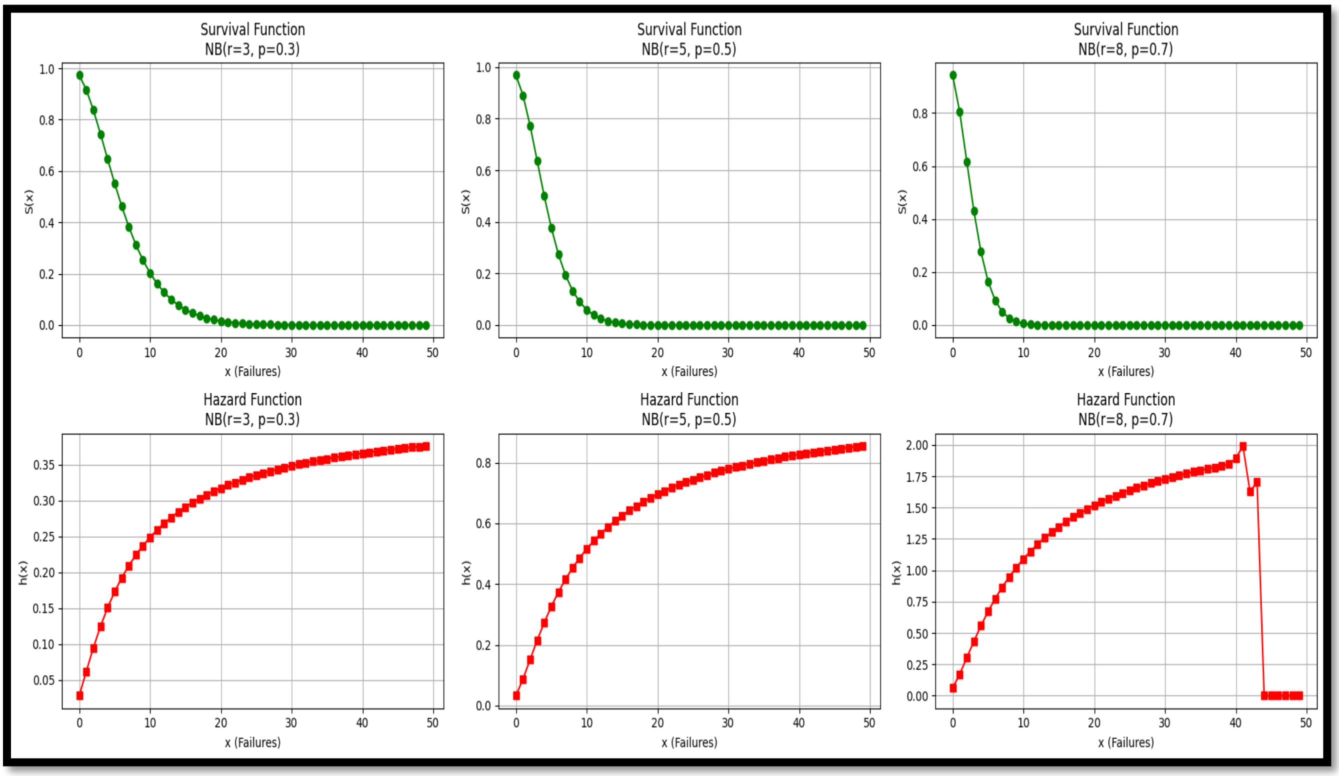


**Figure 3:** *Plot of Survival and Hazard function*

### 3.3 Properties of Negative Binomial

Cumulant Generating Function is used to generate the mean, variance, Skewness and kurtosis

$K_X(t) = -rln(1-(1-p)e^t)$

i. **The mean**

$$\mathrm{E(x)} = \mathrm{K_1} = \mathrm{K'_X(0)}$$

$$\mathrm{K_x(t)} = \frac{d}{dt}[-r\ln(1-(1-p)e^t)]$$

$$= \frac{r(1-p)e^t}{1-(1-p)e^t}$$

$$\mathrm{E(X)} = \mathrm{K'_X(0)}$$

$$= \frac{r(1-p)}{p} \tag{15}$$

ii. The variance

$$Var(x) = K_2 = K''_X(0)$$

$$\mathrm{K''_X(t)} = \frac{d}{dt}\left(\frac{r(1-p)e^t}{1-(1-p)e^t}\right)$$

$$= \frac{r(1-p)e^t[1-(1-p)e^t] + (1-p)e^{2t}}{(1-(1-p)e^t)^2}$$

$$\mathrm{K_2} = \mathrm{K''_X(0)}$$

$$\mathrm{Var(x)} = \frac{r(1-p)}{p^2} \tag{16}$$

**iii. The Skewness**

$$K_3 = K'''_X(0) = \frac{r(1-p)(1+p)}{p^3} \tag{17}$$

**iv. The kurtosis**

$$K_4 = K'^{v}_X(0) = \frac{r(1-p)(1+4p+p^2)}{p^4} \tag{18}$$

**v. The Mode**

The mode indicates the most likely number of failures before achieving a specified number of successes in a series of Bernoulli trials.

The mode is obtained through the ration of successive probabilities:

$$\frac{P(X = x+1)}{(X = x)} = \frac{(x+r)}{(x+1)}(1-p) \tag{19}$$

Set the ratio above $> 1$ to find where PMF is increasing, and $< 1$where it's decreasing.

Now:

$$\frac{(x+r)}{(x+1)}(1-p) > 1$$

$$x < \frac{r(1-p)-1}{p}$$

$$Mode(X) = \begin{cases} \left\lfloor \dfrac{(r-1)(1-p)}{p} \right\rfloor & ,r \geq 1 \\ 0, & otherwise \end{cases} \tag{20}$$

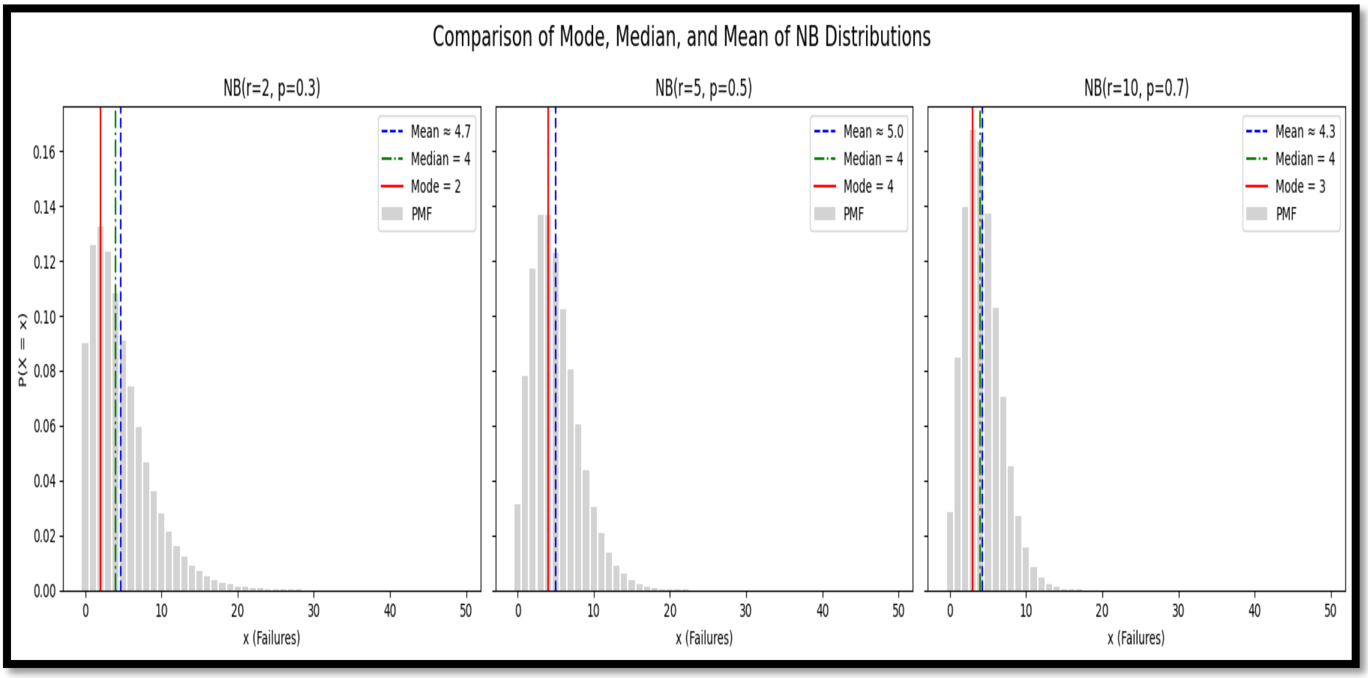


**Table 4**: *Plot of Comparison of Mode, Median and Mean of Negative Binomial*

### 3.4 Asymptotic Properties of Negative Binomial

1. **Convergence to Poisson Distribution**

When $r \rightarrow \infty$ and $p \rightarrow 1$ in such a way that $\mu = \frac{r(1-p)}{p}$ remains fixed, the distribution converges to a Poisson distribution. This is a classic Poisson-Gamma mixture limit.

$\mathbf{NB}(\mathrm{r}, \mathrm{p}) \rightarrow \mathrm{Poisson}(\mu)$

Poisson regression is often a limiting case of NB regression.

2. **Convergence to Normal Distribution**

As $r \rightarrow \infty$, due to the Central Limit Theorem (CLT), the standardized **NB** distribution approaches a Normal distribution:

$$\frac{\mathrm{X} - \mu}{\sigma} \rightarrow \mathrm{N}(0, 1)$$

This holds because the NB distribution can be viewed as the sum of r independent geometrically distributed random variables.

$$X = \sum_{i=1}^{r} G_i, \qquad G_i \sim Geometric(p)$$

For large r, the CLT implies:

$$X = N\left(\frac{r(1-p)}{p}, \frac{r(1-p)}{p^2}\right)$$

This asymptotic normality justifies inference procedures based on normal approximations for large sample sizes.

### 3.5 Practical Applications of the Negative Binomial Distribution

**1. Epidemiology: Modelling Infectious Disease Counts**

The Negative Binomial distribution is widely used in epidemiology to model over dispersed count data, where the variance exceeds the mean a common feature of disease incidence data. For example, suppose a study observes the number of secondary infections caused by an individual infected with a virus. If the average number of secondary cases is 2, but the variance in cases is 6 (indicating clustering of infections, or "super spreading"), a Poisson model would underestimate this variability. Instead, a Negative Binomial model with parameters $r = 2$ and $p = 0.5$ can better capture the probability distribution of secondary cases. Under this model, the probability of an individual causing exactly 5 secondary cases is:

$$P(X = 5) = \binom{5 + 2 - 1}{5} (0.5)^5 (0.5)^2 = 0.0469$$

This illustrates how the Negative Binomial accounts for both average and over dispersed spread patterns, which are crucial for understanding epidemic dynamics and intervention planning.

**2. Insurance and Actuarial Science: Claim Frequency Modelling**

In actuarial science, the Negative Binomial distribution models the frequency of insurance claims when policyholders are heterogeneous. For instance, consider an insurance company covering motor vehicles where the expected number of claims per year is 1.5, but some drivers are much riskier than others, leading to a variance of 3. This overdispersion can be captured by a Negative Binomial distribution with parameters$r = 3, p = 0.67$. The probability of a policyholder making exactly 4 claims in a year is:

$$P(X = 4) = \binom{4 + 3 - 1}{5} (0.33)^4 (0.67)^3 = 0.0536$$

Thus, there is about a 5.36% chance of observing four claims in a year. This example shows how the Negative Binomial distribution allows insurers to more accurately set premiums and reserves by accounting for variability in claim frequency.

**3. Ecology: Species Abundance Modelling**

The Negative Binomial distribution is also used to describe species abundance in ecological studies, particularly where individuals are clustered rather than randomly distributed. Suppose an ecologist surveys plots of land to count the number of a particular insect species. The mean abundance per plot is 3, but the observed variance is 9, indicating aggregation. This overdispersion can be modelled using a Negative Binomial distribution with $r = 2, p = 0.4$. The probability of finding exactly 5 insects in a plot is:

$$P(X = 5) = \binom{5 + 2 - 1}{5}(0.6)^5(0.4)^2 = 0.0746.$$

Hence, there is about a 7.46% probability of encountering exactly five insects. This enables ecologists to quantify species distribution patterns more realistically than under Poisson models, which assume independence.

## 4. CONCLUSION

This study comprehensively examines the convolution of $i.i.d.$ geometric distributions, establishing its relationship with the negative binomial distribution and elucidating its statistical properties and practical applications. The derivation of the PMF and CDF confirms the negative binomial as the sum of r geometric random variables, providing a robust framework for modeling multi-stage "success" processes. Key properties, including moments, generating functions, and asymptotic behavior, highlight the distribution's flexibility in capturing over-dispersion, making it a vital tool in fields such as epidemiology, actuarial science, and ecology.

**Conflict of interest**: The authors declare that they have no conflict of interest among them.

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