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Unveiling the Damped Quantum Harmonic Oscillator

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Abstract

This article dis cusses the often-overlooked "damped" quantum harmonic oscillator, a vibrating system that loses energy over time. We bridge the classical-quantum divide, starting with the familiar equation of motion for a damped oscillator using Hooke's law. Delving into quantum mechanics, we explore how the Schrödinger equation governs its behavior. We then chart a path to understanding its energy changes and time evolution using mathematical tools like annihilation and creation operators, eigenstates, and eigenvalues. We then step through the understanding of its energy changes and time evolution using mathematical tools like annihilation and creation operators, eigenstates, and eigenvalues. Finally, we introduce the time-dependent Schrödinger equation for a damped quantum harmonic oscillator, which paves the way for stable oscillations

Keywords: Damped Quantum Oscillator; Canonical Quantization; Invariant Operator; Time-Dependent Schrödinger Equation

Abstrak

Artikel ini membahas osilator harmonik kuantum teredam, sebuah sistem getaran yang kehilangan energi seiring waktu, yang sering kali diabaikan dalam kajian fisika. Kami menjembatani kesenjangan antara mekanika klasik dan kuantum, dimulai dengan persamaan gerak osilator teredam berdasarkan hukum Hooke. Dalam ranah mekanika kuantum, kami mengeksplorasi bagaimana persamaan Schrödinger mengatur perilaku sistem ini. Selanjutnya, kami menelusuri perubahan energi dan evolusi waktu osilator ini menggunakan alat matematika seperti operator annihilasi dan kreasi, eigenstate, dan eigenvalue. Terakhir, kami memperkenalkan persamaan Schrödinger bergantung waktu untuk osilator harmonik kuantum teredam, yang membuka wawasan terhadap osilasi stabil dalam sistem ini.

Kata kunci: Osilator Kuantum Teredam; Kuantisasi Kanonik; Operator Invarian; Persamaan Schrödinger Bergantung Waktu

I. INTRODUCTION

In classical mechanics, the motion of an object oscillating on a spring is described by Hooke's law(1)(2)(3). This principle states that the force F acting on the object is directly proportional to its displacement x from its equilibrium position and acts(4).

When damping is introduced, the system is called a damped harmonic oscillator(5). Here, an additional force, often due to friction or resistance, causes the oscillations to gradually lose energy over time(6). As a result, the amplitude of the oscillations decreases until they eventually come to a stop(7).

In the realm of quantum mechanics, the harmonic oscillator is a fundamental model used to describe a wide range of physical systems(1). Although the potential energy in quantum systems is not always perfectly harmonic, many can be approximated by a harmonic potential near their stable equilibrium points(8). In quantum mechanics, all oscillators are subject to some form of damping, though this damping is often minimal(9). Unlike in classical mechanics, a quantum oscillator does not lose all its energy but rather reaches a steady state where it continues to oscillate with a reduced amplitude(10).

Unlike classical harmonic oscillators, quantum harmonic oscillators exhibit unique behavior(11).

Even when damping is present, a quantum oscillator does not lose all its energy(12). Instead, it reaches a steady state where it continues to oscillate with a reduced amplitude(13). This phenomenon arises from the principles of quantum mechanics, where energy levels are quantized, meaning they can only take on certain discrete values(14). As a result, the quantum oscillator settles into its lowest energy state, known as the ground state, and remains there, maintaining a stable oscillation despite the damping(15).

The damped quantum harmonic oscillator presents a significant challenge due to the complexity of its description(16). It requires more intricate differential equations than those used for ideal harmonic systems. On the other hand, the practical applications of the quantum damped harmonic oscillator are limited. They are primarily found in microscopic systems, such as vibrating molecules(17). Moreover, experimental studies in this area are difficult. This is due to the need for precise measurements and specialized equipment to maintain controlled environments. This complexity, together with its limited applications, is probably a major reason why many textbooks on quantum mechanics do not cover quantum damped harmonic oscillation in detail.

Despite the challenges, the study of quantum damped harmonic oscillations is intriguing, particularly for educational purposes. It provides insights into why, in classical mechanics, the amplitude of a damped oscillator diminishes over time until it stops, whereas in quantum mechanics, the amplitude tends to stabilize at a certain level. This difference highlights the unique nature of quantum systems and their behavior(18).

This article aims to bridge the gap in understanding by exploring the concept of quantum damped harmonic oscillators in detail. We will begin with a review of the classical harmonic oscillator and its damped counterpart, then move on to discuss the quantum mechanical description and implications of damping in quantum systems. Through this exploration, we hope to shed light on the fascinating differences between classical and quantum oscillatory behavior and provide a comprehensive understanding of quantum damped harmonic oscillations.

II. METHOD

This study is a theoretical research project aimed at exploring the damped quantum harmonic oscillator by bridging classical and quantum mechanics. First, we derive the classical equation of motion for a damped harmonic oscillator and express it in terms of Lagrangian and Hamiltonian formalisms. We then transition to the quantum mechanical description by applying canonical quantization and defining annihilation and creation operators. The Hamiltonian is expressed using these operators, enabling us to determine the eigenstates and eigenvalues. To solve the time-dependent Schrödinger equation, we employ the invariant operator method, introducing a trial invariant operator and transforming it with a unitary operator. This allows us to derive the time-dependent wave function, providing a comprehensive description of the quantum state's evolution. The methodology highlights the intricate connections between classical and quantum damping, offering insights into the unique behavior of quantum oscillatory systems.

III. RESULTS AND DISCUSSION

Damped Harmonic Oscillator in Classical Mechanics

The equation of motion for a damped harmonic oscillator in classical mechanics is given by

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0 \quad (1)$$

In the context where x signifies the deviation of the oscillator from its equilibrium position, the process of canonical quantization necessitates the presence of a Lagrangian denoted as L, which serves as the basis for deriving Equation 1 as the Lagrange equation of motion. To achieve this objective, it is essential to establish a classical mechanics framework utilizing the Rayleigh dissipation function(19).The associated Lagrangian has previously been established to a certain extent and is expressed in relation to the variable x along with an additional variable y, the importance of which will soon become apparent(20). The Lagrangian equation is thus represented as

$$\mathrm{L} = \mathrm{m}\dot{\mathrm{x}}\dot{\mathrm{y}} + \frac{1}{2}\mathrm{R}(\mathrm{x}\dot{\mathrm{y}} - \dot{\mathrm{x}}) - \mathrm{kxy} \tag{2}$$

The equation that y satisfies is derived by altering x,

$$m\ddot{y} - R\dot{y} + ky = 0 \tag{3}$$

The equation is the inverse of Eq. 1 for (x), meaning the solution for (y) increases rapidly over time as the solution for x decreases. Consequently, bilinear structures such as the Hamiltonian (H) can become independent of time. The derivation of the Hamiltonian follows the standard procedure. The canonical momenta are then defined as follows:

$$\begin{aligned} \mathrm{p_x} &= \frac{\partial \mathrm{L}}{\partial \dot{\mathrm{x}}} = \mathrm{m}\dot{\mathrm{y}} - \frac{1}{2}\mathrm{Ry} \\ p_y &= \frac{\partial L}{\partial \dot{y}} = m\dot{x} + \frac{1}{2}Rx \end{aligned} \tag{4}$$

The hamiltoian H reads:

$$\begin{aligned} \mathrm{H} &= \mathrm{p_x}\dot{\mathrm{x}} + \mathrm{p_y}\dot{\mathrm{y}} - \mathrm{L} \\ &= \frac{1}{\mathrm{m}}\mathrm{p_x}\mathrm{p_y} + \frac{\mathrm{R}}{2\mathrm{m}}\mathrm{ypy} - \frac{\mathrm{k} - \mathrm{R}^2}{2\mathrm{m}}\mathrm{xy} \\ &= \frac{1}{\mathrm{m}}\mathrm{p_x}\mathrm{p_y} + \frac{\mathrm{R}}{\mathrm{m}}\mathrm{p_y x} - \mathrm{L} \end{aligned} \tag{5}$$

Now, let's group related terms:

$$\begin{aligned} H &= \left(\frac{R}{m}p_y - \frac{1}{2m}p_x\right)x + \left(\frac{1}{m}p_x + \frac{R}{2m}y\right)p_y - \frac{1}{2m}p_y^2 - \frac{k}{2}x^2 \\ &= \dot{x}x + \dot{y}p_y - \frac{1}{2m}p_y^2 - \frac{k}{2}x^2 \\ &= m\dot{x}\dot{y} - \frac{1}{2m}p_y^2 + kxy \end{aligned} \tag{6}$$

Thus, we have successfully derived Eq. 6 from the initial expression of the Hamiltonian.

The fundamental principle is that according to the equation of motion, the Hamiltonian (H) remains independent of time(21). This fact can be confirmed directly by employing the solutions derived from Eq. 1 and 3, and considering the structure of (H):

$$H = m\dot{x}\dot{y} + kxy \tag{7}$$

For the underdamped case with an initial unit amplitude, H is found to be $k - \frac{R^2}{4m}$.

Quantization, Eigenstates and Eigenvalues

The assumption made is that the coordinates (x) and (y), along with their canonical momenta, follow the Heisenberg uncertainty principle, which states that their uncertainties are related by ($\Delta x \cdot \Delta p \geq \frac{\hbar}{2}$).

$$[x, y] = 0 = [p_x, p_y] \tag{8}$$

This analysis describes how the commutators of the position operators (x) and (y), as well as the momentum operators (p_x) and (p_y) , form the basis of understanding in quantum mechanics(12). First, because ($[x, y] = 0$) and ($[p_x, p_y] = 0$), it means that the position and momentum operators in

this system are commutative, allowing us to measure both simultaneously without altering the results. Next, from the above relationships, we can write:

$$[x, y] = xy - yx = 0 \quad \text{and} \quad [p_x, p_y] = p_x p_y - p_y p_x = 0$$

This indicates that the position operators (x) and (y) commute with each other, as do the momentum operators $(p_x)$and (p_y) . Therefore, we can measure both pairs simultaneously without altering the outcomes.
Further, we can perform a commutator transformation:

$$\begin{aligned} [x, y] &= xy - yx = 0 \\ xy &= yx \end{aligned} \tag{9}$$

$$\begin{aligned} [p_x, p_y] &= p_x p_y - p_y p_x = 0 \\ p_x p_y &= p_y p_x \end{aligned} \tag{10}$$

From the above results, we observe that the position operators (x) and (y) can be applied in different orders with the same result, as can the momentum operators (p_x) and (p_y) .
However, if we change the order of the position and momentum operators, we obtain:

$$\begin{aligned} &[p_x, x] = p_x x - x p_x \\ &= (x p_x + [x, p_x]) - x p_x \\ &= [x, p_x] \\ &= -i \end{aligned} \tag{11}$$

and

$$\begin{aligned} &[p_y, y] = p_y y - y p_y \\ &= (y p_y + [y, p_y]) - y p_y \\ &= [y, p_y] \\ &= -i \end{aligned} \tag{12}$$

then we get the formula,

$$[p_x, x] = [p_y, y] = \frac{\hbar}{i} \tag{13}$$

If we substitute (p_x) and (p_y) according to Eq. 4, we obtain:

$$[\dot{y}, x] = [\dot{x}, y] = \frac{\hbar}{im} \tag{14}$$

The consistency with Eq. 9 is maintained, as mentioned earlier. By employing the Hamiltonian H as defined in Equation 5, we can confirm that the quantum Hamiltonian equations of motion correspond exactly to Eq. 1 and 3. It's worth highlighting once more that the ability to establish time-independent commutation rules like Eq. 9 and 10 relies on the bilinear nature of the variables (x) or (y) and their time-reversal counterparts, ensuring that the decay of x is counteracted by the growth of (y) or (p_x).

$$a = \frac{1}{\sqrt{2\hbar\Omega}}\left(\frac{p_x}{\sqrt{m}} - i\sqrt{m}\Omega x\right), \quad b = \frac{1}{\sqrt{2\hbar\omega}}\left(\frac{p_y}{\sqrt{m}} - i\sqrt{m}\Omega y\right) \tag{15}$$

Let's start by calculating the commutator $[a, a^\dagger]$ to confirm that a and $a^\dagger$ are the correct annihilation and creation operators for the quantum harmonic oscillator. In this case, $a^\dagger$ is the hermitian conjugate of a, given by $[a^\dagger = \frac{1}{\sqrt{2\hbar\Omega}}\left(\frac{p_x}{\sqrt{m}} + i\sqrt{m}\Omega x\right)]$
Let's calculate:

$$[a, a^\dagger] = \left(\frac{1}{\sqrt{2\hbar\Omega}}\left(\frac{p_x}{\sqrt{m}} - i\sqrt{m}\Omega x\right)\right)\left(\frac{1}{\sqrt{2\hbar\Omega}}\left(\frac{p_x}{\sqrt{m}} + i\sqrt{m}\Omega x\right)\right) - \left(\frac{1}{\sqrt{2\hbar\Omega}}\left(\frac{p_x}{\sqrt{m}} + i\sqrt{m}\Omega x\right)\right)\left(\frac{1}{\sqrt{2\hbar\Omega}}\left(\frac{p_x}{\sqrt{m}} - i\sqrt{m}\Omega x\right)\right) \tag{16}$$

After algebraic calculation, we obtain $[a, a^\dagger] = 1$. From here, we can also derive the commutator$[b, b^\dagger]$and demonstrate that b and $b^\dagger$ have the same properties as a and $a^\dagger$.
This confirms that a and b are the appropriate annihilation and creation operators for the quantum harmonic oscillator, where

$$\Omega^2 = \frac{k}{m} - \frac{R^2}{4m^2} \tag{17}$$

and

$$[a, a^+] = [b, b^+] = 1 \quad and \quad [a, b] = [a^+, b^+] = 0 \tag{18}$$

with (a) and (b) and their adjoints (a^+)and (b^+), where

$$H_0 \equiv \frac{1}{m}p_x p_y + m\Omega^2 xy \tag{19}$$

becomes,

$$H_0 \equiv \hbar\Omega(a^+ b + b^+ a) \tag{20}$$

This shows further linear transformation:

$$a = \frac{1}{\sqrt{2}}(A + B), \quad b = \frac{1}{\sqrt{2}}(A - B) \tag{21}$$

With the consequence

$$H_0 = \hbar\Omega(A + A^+ - B - B^+) \tag{22}$$

It's important to mention that (A) and (B) follow the identical commutation rules as (a) and (b), regarding their original variables.

$$A = \frac{a + b}{\sqrt{2}}, \quad B = \frac{a - b}{\sqrt{2}} \tag{23}$$

which from Eq. 10 involves symmetric and anti-symmetric combinations of variables (x) and (y), for example:

$$A = \frac{1}{2\sqrt{\hbar\Omega}}(mp_x + py - im\Omega x + y)$$

(A) changes to even under the transformation ($x - y$) , while (B) changes to odd. (H) can be

expressed in terms of (A) and (B):

$$H = H_0 + H_1 \tag{24}$$

where

$$H_1 = i\frac{\Gamma}{2}(A^+B^+ - AB) \tag{25}$$

By substituting (A) and (B) into the equation for (H_1) , we get:

$$H_1 = i\frac{\Gamma}{2}\left(\frac{1}{2}(a^+ + b^+)(a - b) - \frac{1}{2}(a + b)(a^+ - b^+)\right) \tag{26}$$

Simplifying the equation, we obtain:

$$H_1 = i\frac{\Gamma}{4}(a^+a - a^+b - b^+a + b^+b - a^+a + a^+b + b^+a - b^+b) \tag{27}$$

Some terms cancel out, and we can simplify further to:

$$\begin{aligned} H_1 &= i\frac{\Gamma}{4}(-a^+b - b^+a + a^+b + b^+a) \\ H_1 &= i\frac{\Gamma}{2}(a^+b - b^+a) \end{aligned} \tag{28}$$

Now, let's compare (H_1) with the general form of the Hamiltonian for a harmonic oscillator, $H_0 = \hbar\Omega(a^+b + b^+a)$. We can conclude that:

$$\frac{\Gamma}{2} = \hbar \tag{29}$$

From here, we rearrange the equation to find the value of (Γ), which gives:

$$\Gamma = 2\hbar\Omega \tag{30}$$

In certain physical contexts, (Γ) is often used to describe the frequency or characteristic rate of interaction in a system. In this case, ($\Gamma \equiv \frac{\hbar R}{m}$) reveals the relationship between this frequency and the relevant constant (R) with respect to the system's mass (m) . so that

$$\Gamma \equiv \frac{\hbar R}{m} \tag{31}$$

The decay constant for the classical variable (x) is ($\frac{\Gamma}{2\hbar}$). In the limit ($R \to 0$), $[H \to \hbar\omega(A + A^+ - B - B^+)]$

To understand the time evolution of the damped harmonic oscillator state, we must determine its Hamiltonian's eigenstates. This process involves recognizing that (H_1), along with two other operators, forms an algebra. These operators are defined as(2):

$$\begin{aligned} X &= \frac{1}{2}(A^+B^+ + AB) \\ Y &= \frac{1}{2}(A^+B^+ - AB) \\ Z &= \frac{1}{2}(A^+A + BB^+) \end{aligned} \tag{32}$$

(H_1) is proportional to (Y). Their commutator relations are

$$\begin{aligned} [X, Y] &= iZ \\ [Z, Y] &= iX \\ [Z, X] &= iY \end{aligned} \tag{33}$$

The operator (H_0) is in a commutative relationship with X, Y, and Z and functions as the Casimir operator within this algebraic structure.We can use these commutator relationships to express X^2, Y^2, and Z^2 in terms of $H_1$and H_0:

$$\begin{aligned} X^2 &= \frac{1}{c^2}H_1^2 \\ Y^2 &= 1 \\ Z^2 &= c^2 H_1^2 \end{aligned} \tag{34}$$

then we substitute these values into the formula $Z^2 - X^2 - Y^2 = H_0^2 - \frac{1}{4}$:

$$c^2 H_1^2 - \frac{1}{c^2}H_1^2 - 1 = H_0^2 - \frac{1}{4} \tag{35}$$

when simplified, we get:

$$c^2 - 1 = H_0^2 - \frac{1}{4} \tag{36}$$

then, because $H_1 = cY$, we replace H_1^2 with c^2Y^2:

$$c^4 Y^2 - \frac{1}{c^2}c^2 Y^2 - 1 = H_0^2 - \frac{1}{4} \tag{37}$$

the result is:

$$c^2 Y^2 - Y^2 - 1 = H_0^2 - \frac{1}{4} \tag{38}$$

Or it can be further simplified to:

$$\begin{aligned} c^2 - 1 &= H_0^2 - \frac{1}{4} \\ Z^2 - X^2 - Y^2 &= {H_0}^2 - \frac{1}{4} \end{aligned} \tag{39}$$

where,

$$H_0 \equiv 2\hbar\Omega h_0 \tag{40}$$

so that

$$h_0 = \frac{1}{2}(A^+A - B^+B) \tag{41}$$

The algebra described in Eq. 30-32 is sometimes denoted as QU(2) or O(2,1). While the angular momentum operators (L_x), (L_y), (L_z) satisfy Eq. 30, the corresponding equation analogous to Eq. 31 has a negative sign on the right-hand side. For instance, $[L_z, L_y] = -iL_x$. In analyzing the QU(2) algebra, states are usually labeled by the eigenvalues of (h_0), $(\frac{1}{2}(4n_A - n_B) = j)$ and with the eigenvalues of $(Z - \frac{1}{2})$, $(m = \frac{1}{2}(n_A + n_B))$. The non-compact nature of this algebra is shown by the inequality

$$\frac{1}{2}(n_A + n_B) \geq \frac{1}{2}(n_A - n_B), \quad m \geq |j| \tag{42}$$

However, in the current context, we are more interested in the eigenvalues of (Y) compared to the eigenvalues of (Z) . To achieve this, we use the relation

$$[e^{\mu X} Y e^{-\mu X} = Y\cos\mu + iZ\sin\mu] \quad (43)$$

which becomes

$$[e^{(\pi/2)X} Y e^{-(\pi/2)X} = iZ] \quad (44)$$

or equivalently

$$Y = ie^{-(\pi/2)X} Z e^{(\pi/2)X} \quad (45)$$

Let the eigenstate of (Z) be denoted by ($|j,m>$) with eigenvalue ($\frac{1}{2}(n_A + n_B + 1) = m + \frac{1}{2}$).Then the eigenstate of \(Y \) is given by $e^{-(\pi/2)X}\ |j,\ m>$:

$$Y e^{-(\pi/2)X}\ |j,m> = i\left(m + \frac{1}{2}\right)|j,m>\Big) \quad (46)$$

There exists a set of eigenstates and their corresponding eigenvalues

$$e^{-(\pi/2)X} Y e^{(\pi/2)X} = iZ \quad (47)$$

Their eigenstates are $e^{-(\pi/2)X}\ |j,m>$:

$$Y e^{-(\pi/2)X}\ |j,m> = -i\left(m + \frac{1}{2} e^{(\pi/2)X}\right)|j,m>\Big)$$

The eigenvalues in both cases are purely imaginary. Furthermore, it can be demonstrated that (m) has a minimum value, which is the same for the eigenstates from Eq. 36 and 38. This can be demonstrated using the traditional method as outlined in reference. The "lowering" operator is $(X + Z)$ as can be directly observed from Eq. 30 and 31, which gives

$$[(X + Z), Y] = i(X + Z) \quad (48)$$

If (ψ_{im}) is the eigenstate of (Y) with eigenvalue $i\left(m + \frac{1}{2}\right)$), ($m > 0$), then

$$Y(X + Z)\psi_{im} = i\left(m + \frac{1}{2}\right)(X + Z)\psi_{im} \quad (49)$$

When (m) is reduced to one state eigenstate of (Y) with the least value of (m) is (M). Then it becomes

$$(X + Z)\psi_{im} = 0 \quad (50)$$

By multiplying $(Z - X)$ and using Eq. 32 and 33, the equation is obtained

$$[M^2 = j^2] \quad (51)$$

The lowest possible value for (m) is ($|j|$). The "raising" operator is ($X - Z$). When applied to ($\psi_{im} > 0$), the set of eigenstates satisfying Eq. 32 with an eigenvalue $(-i\left(m + \frac{1}{2}\right))$ results in the eigenstate of (Y) with eigenvalue $(-i\left(m + \frac{1}{2}\right) + i) or (-i\left(m - \frac{1}{2}\right))$. The minimum value of \(m \) for this set of negative imaginary eigenvalues is also ($|j|$).

In essence, there exist two distinct sets of eigenstates: one exhibiting positive imaginary eigenvalues $(i\left(m + \frac{1}{2}\right))$, and the other featuring negative imaginary eigenvalues $(-i\left(m + \frac{1}{2}\right))$. In both scenarios, (m) is equal to $(\frac{1}{2}(n_A + n_B))$.The positive imaginary eigenvalues for positive (j) are $(i\left(j + \frac{1}{2}\right))$, $(i\left(j + \frac{3}{2}\right))$, $(i\left(j + \frac{5}{2}\right))$, and so on, where $(j = \frac{1}{2}(n_A + n_B))$. Conversely, the negative ones are

$(-i\left(j+\frac{1}{2}\right))$ and subsequent values, forming a negative eigenvalue spectrum for positive (j).
Therefore, the eigenvalues of (H) are given by

$$2\hbar\Omega j \pm i\left(m+\frac{1}{2}\right)\Gamma = \hbar\Omega(n_A - n_B) \pm i\Gamma(2n_A + n_B + 1) \tag{52}$$

Eigenstates with negative eigenvalues represent decaying states, indicating a decrease in the corresponding physical quantities over time. Conversely, eigenstates with positive eigenvalues represent growing states, indicating an increase in the associated physical quantities. As anticipated from these observations, the set of states described by Eq. 38 represents the time reverse of the states given by Eq. 36.
To elaborate further, when we consider time reversal, we observe that under this transformation, the eigenstates with negative eigenvalues in Eq.38 correspond to eigenstates with positive eigenvalues in Eq. 36, and vice versa. This symmetry in the eigenstates under time reversal highlights the dual nature of the system's behavior, where decaying states transform into gro

$$\begin{matrix}[A \leftrightarrow -A^{+}] \\ [B \leftrightarrow -B^{+}]\end{matrix} \tag{53}$$

Subsequently, (H_0), (X), and (Z) exhibit even behavior under time reversal, whereas (Y) demonstrates odd behavior. Equation 38 is essentially the time-reversed version of the equation preceding Eq. 35. Upon applying the time reversal operation to Eq. 38 (indicated by the superscript T), we obtain

$$[Y\left[e^{-(\pi/2)X}\right|j,m>]^{T} = i\left(m+\frac{1}{2}\right)\left[e^{-(\pi/2)X}\right|j,m>]^{T}] \tag{54}$$

Compared to Eq. 38

$$e\text{^}\{(\pi/2)\}\, X\, |j,m>= \left[e^{-(\pi/2)X}\right|j,m>]^{T} \tag{55}$$

Time Dependence in a Damped Harmonic Oscillator

The classical equation of motion for a damped harmonic oscillator with a general friction coefficient is shown in Eq. 1. We then consider the mass of the particle set to unity and only consider the special case where the friction coefficient and the system frequency decrease rationally. In this case, $(\gamma(t))$ and $(w(t))$in the given equation become

$$\gamma(t) = \frac{\gamma_0}{1+qt}, \quad w(t) = \frac{w_0}{1+qt} \tag{56}$$

assuming $(q > 0)$ and (γ_0, w_0) are positive constants.
The corresponding quantum Hamiltonian for this equation is

$$H = \frac{p^2}{2\Gamma(t)} + \frac{1}{2}\Gamma(t)w^2(t)x^2 \tag{57}$$

with $[x,p] = i$ we set $\hbar = 1$ and

$$\Gamma(t) = e^{\int_0^t \gamma(s)ds} = (1+qt)^{\gamma_0/q} \tag{58}$$

The Hamiltonian can be rewritten as

$$H = \frac{1}{\Gamma(t)}J_- + \Gamma(t)w^2(t)J_+ \tag{59}$$

where

$$J_+ = \frac{1}{2}x^2, \quad J_- = \frac{1}{2}p^2 \tag{60}$$

By introducing an additional operator

$$J_0 = \frac{i}{4}(px + xp) \tag{61}$$

we form the su(2) algebra defined by

$$[J_+, J_-] = 2J_0, \quad [J_0, J_\pm] = \pm J_\pm \tag{62}$$

where we have $J_\pm^\dagger = J_\pm, J_0^\dagger = -J_0$. The Hamiltonian that generates this equation is not unique. By setting $H = H_-(t)J_- - iH_0(t)J_0 + H_+(t)J_+$, we obtain this equation if the following relations are satisfied:

$$\dot{H}_- = -\gamma(t)H_-, \quad H_-H_+ - \frac{1}{4}H_0^2 - \frac{1}{2}\dot{H}_0 - \frac{1}{2}\gamma(t)H_0 = w^2(t) \tag{63}$$

For simplicity, throughout this paper we set $H_0 = 0$ and then we have the previous Hamiltonian.

To obtain the quantum mechanical solution of the time-dependent Hamiltonian system, it is easier to use the invariant operator. To investigate the system mechanically, we introduce the trial invariant operator as

$$I = h_1(t)J_+ + ih_2(t)J_0 + h_3(t)J_- \tag{64}$$

where (h_1, h_2, h_3) are real functions and (I) is Hermitian. The invariant operator (I) satisfies

$$\frac{dI}{dt} = \frac{\partial I}{\partial t} + i[H, I] = 0 \tag{65}$$

By substituting this equation into the invariant equation, we obtain

$$\begin{aligned} \dot{h_1} &= -\Gamma w^2 h_2, \\ \dot{h_2} &= \frac{2}{\Gamma}h_1 - 2\Gamma w^2 h_3, \\ \dot{h_3} &= \frac{1}{\Gamma}h_2. \end{aligned} \tag{66}$$

If we assume a particular form for $h_1(t), h_2(t)$, and $h_3(t)$ as

$$h_1(t) = \alpha(1+qt)^A, \quad h_2(t) = \beta(1+qt)^B, \quad h_3(t) = \gamma(1+qt)^C \tag{67}$$

we obtain the following equations

$$\begin{aligned} &\alpha qA + \beta w_0^2 = 0, \quad \beta - \gamma qC = 0, \quad q\beta B = 2\alpha - 2w_0^2\gamma \\ &A = \frac{\gamma_0}{q} + B - 1, \quad C = B + 1 - \frac{\gamma_0}{q}. \end{aligned} \tag{68}$$

We then transform the invariant equation with the appropriate unitary operator U as

$$I' = UIU^\dagger \tag{69}$$

by choosing U as

$$U = U_2U_1 = e^{k_2(t)J_0}e^{ik_1(t)J_+} \tag{70}$$

By choosing k_1 and $k_2(t)$ as

$$k_1 = -\frac{h_2}{2h_3}, \quad e^{-k_2(t)} = \frac{1}{\gamma}(1+qt)^{-C} \tag{71}$$

we obtain the transformed invariant

$$I' = \frac{p^2}{2} + \frac{1}{2}\left(\alpha - \frac{\beta^2}{4\gamma}\right)\gamma(1+qt)^{A+C}x^2 \tag{72}$$

From the fact that both I and I' are time invariants, we know that $A + C = 0$. If we choose $\gamma = 1$, we get

$$h_1 = w_0^2(1+qt)^{-1+\gamma_0/q}, \quad h_2 = q - \gamma_0, \quad h_3 = (1+qt)^{1-\gamma_0/q} \tag{73}$$

and the transformed invariant becomes

$$I' = \frac{p^2}{2} + \frac{1}{2}w_1^2x^2 \tag{74}$$

where

$$w_1^2 = w_0^2 - \frac{1}{4}(q-\gamma_0)^2 \tag{75}$$

The eigenvalue equation for I' is

$$I'|n,t\rangle' = w_1\left(n+\frac{1}{2}\right)|n,t\rangle', \quad n = 0,1,2, \tag{76}$$

Using the basis $|x\rangle$, we get the wave solution

$$\langle x|n,t\rangle' = \left(\frac{w_1}{\pi}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}H_n\left(\sqrt{w_1}x\right)e^{-\frac{1}{2}w_1x^2} \tag{77}$$

where $H_n(x)$ is the Hermite polynomial of order n. With the transformation $|n,t\rangle = U^\dagger|n,t\rangle'$, we get

$$I|n,t\rangle = w_1\left(n+\frac{1}{2}\right)|n,t\rangle, \quad n = 0,1,2, \tag{78}$$

and the wave function

$$\langle x|n,t\rangle = \left(\frac{w_1w_0^2}{\pi h_1(t)}\right)^{1/4}H_n\left(\sqrt{\frac{w_1w_0^2}{h_1(t)}}x\right)\exp\left[-\frac{w_0^2}{2h_1(t)}\left(w_1 - i\frac{\gamma_0-q}{2w_0^2}h_1(t)\right)x^2\right] \tag{79}$$

The step operator is expressed as

$$\begin{aligned} a(t) &= \sqrt{\frac{w_1h_1}{2w_0^2}}\left(1 - i\frac{h_2}{2w_1}\right)x + i\sqrt{\frac{w_0^2}{2w_1h_1}}p \\ a^\dagger(t) &= \sqrt{\frac{w_1h_1}{2w_0^2}}\left(1 + i\frac{h_2}{2w_1}\right)x - i\sqrt{\frac{w_0^2}{2w_1h_1}}p \end{aligned} \tag{80}$$

and the invariant is expressed as

$$I(t) = \frac{w_1}{2}\left[a\backslash dagger(t)a(t) + \frac{1}{2}\right]. \tag{81}$$

The Fock space representation of the step operator is

$$\begin{aligned} a(t)|n,t\rangle &= \sqrt{n}|n-1,t\rangle, \\ a^{\dagger}(t)|n,t\rangle &= \sqrt{n+1}|n+1,t\rangle. \end{aligned} \tag{82}$$

By expressing the position and momentum operators in terms of the step operator, we get

$$\begin{aligned} x &= \sqrt{\frac{w_0^2}{2w_1h_1}}[a(t) + a^{\dagger}(t)], \\ p &= i\sqrt{\frac{h_1}{2w_0^2w_1}}\left[\left(w_1 + i\frac{h_2}{2}\right)a(t) - \left(w_1 - i\frac{h_2}{2}\right)a^{\dagger}(t)\right]. \end{aligned} \tag{83}$$

The time-dependent wave function for the given Hamiltonian then takes the form

$$|\psi_n(t)\rangle = |n,t\rangle e^{i\gamma_n(t)}, \tag{84}$$

with the phase

$$\gamma_n(t) = -\frac{w_0^2}{qw_1}\left(n + \frac{1}{2}\right)\ln(1+qt) \tag{85}$$

Therefore, the time-dependent wave function for the Hamiltonian is

$$\begin{aligned} \langle x|\psi_n(t)\rangle = \left(\frac{w_1w_0^2}{\pi h_1(t)}\right)^{1/4} H_n\left(\sqrt{\frac{w_1w_0^2}{h_1(t)}}x\right)\exp\left[-\frac{w_0^2}{2h_1(t)}\left(w_1\right.\right. \\ \left.\left. - i\frac{\gamma_0 - q}{2w_0^2}h_1(t)\right)x^2\right] e^{-i\frac{w_0^2}{w_1}(n+1/2)\ln(1+qt)} \end{aligned} \tag{86}$$

IV. CONCLUSION

In this article, we explored the damped quantum harmonic oscillator, comparing classical and quantum mechanics. We reviewed how damping causes energy loss in classical systems, then examined the quantum case using the Schrödinger equation. By deriving the Hamiltonian and using annihilation and creation operators, we analyzed the system's eigenstates and time evolution. We showed that, unlike classical systems, the quantum oscillator reaches a steady state with reduced amplitude. This study highlights key differences between classical and quantum behaviors, enriching our understanding of quantum mechanics and illustrating its profound principles.

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